<u>Definition 2</u> (Self-adjoint operator , Normal operator and Unitary operator)

Let $T: H \to H$ be a bounded linear operator on Hilbert space H and $T^* : H \to H$ be adjoint operator of T . T is said to be

1- Self-adjoint operator if $T^* = T$.

2- Normal operator if $T^*T = TT^*$

3- Unitary operator if $T^*T = TT^* = I$. that is T is invertible and $T^* = T^{-1}$.

Example 2 :

1- Zero operator 0 and identity operator I on Hilbert space H are self-adjoint , since:

(1) $< 0^*x, x > = < x, 0x > = < x, 0 >= 0 = < 0, x > = < 0x, x >$

$\therefore\ 0^*x = 0x \quad \forall x \in H$, therefor $0^* = 0$

(2) $< I^*x, x > = < x, Ix > = < x, x >= < Ix, x >$

$\therefore\ I^*x = Ix \quad \forall x \in H$, therefor $I^* = I$

2- Zero operator 0 and identity operator I on Hilbert space H are normal operator , since : H.W.

3- Zero operator 0 and identity operator I on Hilbert space H are unitary operator , since : H.W.

<u>Theorem 2</u>

Let $T : H \to H$ be a bounded linear operator where H is Hilbert space , then

1- if T is self-adjoint operator, then T is normal operator .

2- if T is unitary operator , then T is normal operator .

Proof :

1- let T is self-adjoint operator

$\therefore T^* = T$

We must prove that T is normal operator, i.e. to prove $T^*T = TT^*$

$< T^*Tx, y > = < TT^*x, y >$ (since $T^* = T$)

$\therefore TT^*x = T^*Tx \quad \forall x \in H$

$\therefore TT^* = T^*T$

$\therefore$ T is normal operator.

2- let T is unitary operator,

$\therefore T^*T = TT^* = I$, that is $T^*T = TT^*$

$\therefore$ T is normal operator

Note the convers of this theorem is not true . show that. H.W.

<u>Theorem 2 :</u>

The product of two bounded self-adjoint operator T_1 and T_2 on Hilbert space is self-adjoint if and only if $T_1 T_2 = T_2 T_1$.

Proof :

let T_1 and T_2 are self-adjoint , therefor $T_1^* = T_1$ and $T_2^* = T_2$ and let $T_1 T_2 = T_2 T_1$

Now, we must prove that $T_1 T_2$ self-adjoint

$(T_1 T_2)^* = T_2^* T_1^* = T_2 T_1 = T_1 T_2$

$\therefore T_1 T_2$ Is self-adjoint

Conversely :

let T_1, T_2 and $T_1 T_2$ are self-adjoint .

$\therefore\ T_1^* = T_1\ ,\ \ T_2^* = T_2$ and $(T_1T_2)^* = T_1T_2$ …………(1)

We must prove that $T_1\,T_2 = T_2\,T_1$

$\because\ (T_1\,T_2)^* = T_2^*\,T_1^* = T_2\,T_1$ ……………(2)

$\therefore$ by (1)and (2)we get $T_2\,T_1 = T_1\,T_2$. ■

Theorem 3 :

(a) If T_1 and T_2 are two self-adjoint operators over H, then $T_1 + T_2$ is self-adjoint

(b) If T_1 is self- adjoint and α be any real scalar, then $\alpha\,T_1$ is self-adjoint

Proof :

(a) $\because\ T_1, T_2$ are self-adjoint

$\therefore\ T_1^* = T_1\ ,\ \ T_2^* = T_2$

$(T_1 + T_2)^* = T_1^* + T_2^* = T_1 + T_2$

(b) $\because\ T_1$ self-adjoint

$\therefore\ T_1^* = T_1$

$(\alpha\,T_1)^* = \overline{\alpha}\,T_1^* = \overline{\alpha}\,T_1$

$= \alpha\,T_1$ ($\bar{\alpha} = \alpha$ because α is real scalar)

Theorem 4 :

Let T and S are tow normal operator on Hilbert space H if $ST^* = T^*S$ and $TS^* = S^*T$, then

1- $T + S$ is normal .

2- TS is normal .

Proof : H.W.

Lemma

For any bounded linear operator Ton Hilbert space H, if $T_1 = \frac{1}{2}\,(T + T^*)$ and $T_2 = \frac{1}{2i}\,(T - T^*)$, then

1- T_1 and T_2 are self-adjoint

2- $T = T_1 + iT_2$ is unique

3- $T^* = T_1 - iT_2$

4- T is normal iff T_1 and T_2 are commutative . H.W.

Proof:

1- we must prove that T_1 and T_2 self-adjoint

$T_1^* = (\frac{1}{2}\,(T + T^*))^* = \frac{\bar{1}}{2}\,(T + T^*)^*$ [Since $(\lambda\, T)^* = \bar{\lambda}\, T^*$]

$= \frac{1}{2}\,(T^* + (T^*)^*)$ [Since$(T + S)^* = T^* + S^*$]

$= \frac{1}{2}\,(T + T^*) = T_1$

$\therefore\ T_1^* = T_1$

Thus T_1 self-adjoint

$T_2^* = (\frac{1}{2i}\,(T - T^*))^* = \overline{\left(\frac{1}{2i}\right)}(T - (T^*)^*) = \frac{-1}{2i}\,(T^* - T) = \frac{1}{2i}\,(T - T^*) = T_2$

$\therefore\ T_2^* = T_2$

$\therefore\ T_2$ self-adjoint

2- we must prove that $T = T_1 + iT_2$ is unique

Let $S = T_1 + i\,T_2$ we must prove that $S = T$

Now,

$$S = T_1 + iT_2 = \frac{1}{2}(T + T^*) + i\left(\frac{1}{2i}(T - T^*)\right) = \frac{1}{2}T + \frac{1}{2}T^* + \frac{1}{2}T - \frac{1}{2}T^* = \frac{1}{2}T + \frac{1}{2}T = T$$

$\therefore T = T_1 + iT_2$ unique

3- we must prove that $T^* = T_1 - iT_2$

$$T^* = (T_1 + iT_2)^* = T_1^* + (i(T_2))^* = T_1^* + \bar{i}\, T_2^* = T_1^* - i\, T_2^*$$

$\because$ T_1 and T_2 self-adjoint

$\therefore$ $T_1^* = T_1$ and $T_2^* = T_2$

$\therefore$ $T^* = T_1 - iT_2$

$\blacksquare$

Theorem

If T_1 and T_2 are two unitary operator on Hilbert space H , then $T_1 T_2$ is unitary .

Proof :