Chapter Nine

Hilbert – adjoint operator

<u>Definition 1</u> (Hilbert -adjoint operator)

Let $T : H_1 \to H_2$ be a bounded linear operator where H_1 and H_2 are Hilbert space , then the adjoint operator T is denoted by T^* and define by

$$T^* : H_2 \to H_1 \text{ such that } < Tx, y > = < x, T^*y > \text{ for all } x \in H_1 \text{ and } y \in H_2 ,$$

Note :

1- T^* is bounded linear operator
2- $\| T^* \| = \| T \|$
3- T^* is unique

Example 1 :

let $T: R^3 \to R^2$ be a linear bounded operator and defined as :

$$T(x_1 , x_2 , x_3) = (x_2 + 3x_3 , 2x_1)$$

We can compute T^* as following :

Let y= $(y_1 , y_2) \in R^2$, $x = (x_1 , x_2 , x_3) \in R^3$

Now, $< Tx, y > = < x, T^*y >$

$\therefore < T (x_1 , x_2 , x_3), (y_1 , y_2) > = < (x_1 , x_2 , x_3), T^* (y_1 , y_2) >$ ……………………….(1)

$$\because < x , y > = \sum_{i=1}^{n} x_i y_i \quad \forall x , y \in R^n$$

$\therefore < T (x_1 , x_2 , x_3), (y_1 , y_2) > = < (x_2 + 3x_3 , 2x_1) , (y_1 , y_2) > = (x_2 + 3x_3)y_1 + 2x_1y_2$

$$= x_2 y_1 + 3x_3 y_1 + 2x_1y_2 = < (x_1 , x_2 , x_3), (2y_2 , y_1 , 3y_1) >$$

$\therefore$ by (1) we get

$< (x_1 , x_2 , x_3), (2y_2 , y_1 , 3y_1) > = < (x_1 , x_2 , x_3), T^* (y_1 , y_2) >$

$\therefore \quad T^* (y_1 , y_2) = (2y_2 , y_1 , 3y_1)$

H.W.

let $T: R^3 \to R^2$ be a linear bounded operator defined by : $T(x_1, x_2, x_3) = (x_1 + x_2, x_2 - x_3)$

Find T^*.

<u>Theorem 1</u> (Properties of adjoint operator)

Let H_1, H_2 be Hibert space and let $S: H_1 \to H_2$, $T: H_1 \to H_2$ be two bounded linear operators and α any scalar then we have:

(a) $< T^*y , x > = < y, Tx > \quad (x \in H_1 , y \in H_2)$

(b) $(S + T)^* = (S^* + T^*)$

(c) $(\alpha T)^* = \bar{\alpha} T^*$

(d) $(T^*)^* = T$

(e) $\| T^* T \| = \| T T^* \| = \| T \|^2$

(f) $T^* T = 0 \Leftrightarrow T=0$

(g) $(S T)^* = T^* S^*$ $(assuming\ H_2 = H_1)$

Proof

(a) By def. of inner product ($< x , y > = \overline{< x, y >}$)

$\therefore < T^* y , x > = \overline{< x , T^* y >} \quad \forall x \in H_1 , y \in H_2$

$= \overline{< T x , y >} = < y , Tx >$

(b) $\because (< Tx , y > = < x, T^* y >)\ \forall x \in H_1 , y \in H_2$ (by definition)

$\therefore < x, (S + T)^* y > = < (S + T)x, y > = < (Sx + Tx), y > = < Sx, y > + < Tx, y >$

$= < x, S^*y > + < x, T^*y > = < x, S^*y + T^* y > = < x, (S^* + T^*)y >$

$\therefore (S+T)^* y = (S^* + T^*)\, y \quad \forall y \in H_2$

$\therefore (S+T)^* = S^* + T^*$

(c) $\because < T^* y, x > = < y, Tx > \ \forall x \in H_1, y \in H_2$ *(by (a))*

$\therefore < (\alpha T)^* y, x > = < y, (\alpha T)x > = < y, \alpha(Tx) > = \bar{\alpha} < T^* y, x > = < \bar{\alpha}\, T^* y, x >$

$\therefore (\alpha T)^* y = \bar{\alpha}\, T^* y \ \ \forall y \in H_2$ *, therefor* $(\alpha T)^* = \bar{\alpha}\, T^*$

(d) $\because < T^* y, x > = < y, Tx > \ \forall x \in H_1, y \in H_2$

$\therefore$ we can write $< (T^*)^* x, y > = < x, T^* y > = < Tx, y >$

$\therefore (T^*)^* x = Tx \ \ \forall x \in H_1$, therefor $(T^*)^* = T$

(e) At first we prove that $\| T^* T \| = \| T \|^2$

i.e to prove that $\| T^* T \| \leq \| T \|^2$ and $\| T\, T^* \| \geq \| T \|^2$

Now , to prove $\| T^* T \| \geq \| T \|^2$

$\because \| x \| = \sqrt{< x, x >}$

$\therefore \| x \|^2 = < x, x >$

$$\begin{aligned}\| T(x) \|^2 = < Tx, Tx > \ &= < T^* Tx, Tx > \\ &\leq |< T^* Tx, x >| \\ &\leq \| T^* T_x \| \ \| x \| \quad \text{(by Theorem (1.2))} \\ &\leq \| T^* T \| \, \| x \| \, \| x \| \quad \text{(since T}^*\text{T is bounded operator)} \\ &\leq \| T^* T \| \ \| x \|^2\end{aligned}$$

By taking supermum of both sides, we get :

$$sup\ \{ \| Tx \|^2, \| x \| = 1 \} \leq \| T^* T \|$$

$\| T \|^2 \leq \| T^* T \|$ ………………*(1)* (since $\| T \| = sup\ \{ \| Tx \|, \| x \| = 1 \}$)

To prove $\| T \|^2 \geq \| T^*T \|$

$\| T^*T \| \leq \| T^* \| \; \| T \|$

$\leq \| T \| \| T \|$ (since $\| T^* \| = \| T \|$)

$\| T^* T \| \leq \| T \|^2$ ……………*(2)*

By (1) and (2) we have

$\| T^*T \| = \| T \|^2$………………*(3)*

Now, we prove that $\| T T^* \| = \| T \|^2$

In (3) we replacing T by T* then we get $\| T^{**} T^* \| = \| T^* \|^2$

$\| T T^* \| = \| T^* \|^2$ (since $T^{**} = T$)

$\because \| T^* \| = \| T \|$

$\therefore \| T T^* \| = \| T \|^2$ ……………………………*(4)*

Then by (3) and (4) we get

$\| T^* T \| = \| T T^* \| = \| T \|^2$

(f) $T=0 \leftrightarrow \| T \| = 0 \leftrightarrow \| T \|^2 = 0$

$\because by \; (\boldsymbol{e}) \; \| T^* T \| = \| T \|^2 = 0$

$\| T^* T \| = 0 \; \leftrightarrow \; T^*T = 0$

(g) *H.W*

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