Chapter Five

Inner Product Space

<u>Definition :</u> (Inner product space) or (pre- Hilbert space)

An inner product on a vector space X over a field F is a function of two variables $<.,.>: X \times X \to R$ (or C) such that :

1- $< x,x \geq 0 \quad \forall x \in X$
2- $< x,x > = 0 \quad iff \;\; x = 0$
3- $< x,y > = \overline{< y,x >} \quad \forall x,y \in X \quad if \quad X = C.$
$\qquad\qquad = < y,x > \quad \forall x,y \in X \quad if \quad X = R.$
4- $< \alpha x + \beta y, z > = \alpha < x,z > + \beta < y,z > \quad \forall x,y,z \in X \; and \;\; \alpha,\beta \in F.$

$(X, <.,.>)$ is called inner products space or (pre- Hilbert space)

Same time we say that X is inner product space or (pre- Hilbert space) .

<u>Note :</u> The usual inner product on R^n is define by $< x,y > = \sum_{i=1}^{n} x_i \, y_i \;\; , \forall x,y \in R^n$.

<u>Example 1:</u>

Euclidean space R^n is inner product space where $< x,y > = \sum_{i=1}^{n} x_i \, y_i \;\; , \forall x,y \in R^n$.

<u>Solution:</u>

<u>Example 2:</u>

C^n is complex inner product space where $\forall x,y \in C^n, < x,y > = \sum_{i=1}^{n} \overline{x_i} \, y_i$.

<u>Solution:</u>

H.W

1- Show that R^2 is pre- Hilbert space .

2- Is R^3 an inner product space ? why ?

Theorem 1: (The properties of inner product)

Let X be an inner product space if $x, y, z \in X$ and $\alpha, \beta \in C$ then :

(a) $< \alpha x - \beta y, z > = \alpha < x, z > - \beta < y, z >$

(b) $< x, \alpha y + \beta z > = \overline{\alpha} < x, y > + \overline{\beta} < x, z >$

(c) $< x, \alpha y - \beta z > = \overline{\alpha} < x, y > - \overline{\beta} < x, z >$

(d) $< x, 0 > = < 0, x > = 0 \qquad \forall x \in X$

(e) $< x, y + z > = < x, y > + < x, z >$

Proof :

(a)

Remark : Every inner product is norm, and the norm induced by inner product is define by $\| x \| = \sqrt{< x, x >}$

Theorem 2: (The Cauchy Schwarz inequality)

Let x and y be two vectors in an inner product space X then,

$$|< x, y >| \leq \| x \| \, \| y \|$$

Proof:

Let X be an inner product space and let $x, y \in X$

If $x = 0$ or $y = 0 \implies \| x \| = 0$ or $\| y \| = 0$ and $|< x, y >| = 0$

Thus the inequality is hold.

Now, if $x \neq 0$ and $y \neq 0$, let λ be any scalar we have ,

$$< x + \lambda y, x + \lambda y > = < x, x + \lambda y > + < \lambda y, x + \lambda y >$$

$$= < x, x > + < x, \lambda y > + < \lambda y, x > + < \lambda y, \lambda y >$$

$$= < x, x > + \overline{\lambda} < x, y > + \lambda < y, x > + \lambda \overline{\lambda} < y, y >$$

$$= \| x \|^2 + \overline{\lambda} < x, y > + \lambda \overline{< x, y >} + | \lambda |^2 \| y \|$$

$\because y \neq 0 \rightarrow \| y \| \neq 0$

There for we can put $\lambda = -\frac{<x,y>}{\|y\|^2}$, then we get

$$= \| x \|^2 - \frac{\overline{< x, y >}}{\| y \|^2} < x, y > - \frac{< x, y >}{\| y \|^2} \overline{< x, y >} + \left| \frac{-< x, y >}{\| y \|^2} \right|^2 \| y \|^2$$

$$= \| x \|^2 - \frac{|< x, y >|^2}{\| y \|^2} - \frac{|< x, y >|^2}{\| y \|^2} + \frac{|< x, y >|^2}{\| y \|^2}$$

$$= \| x \|^2 - 2 \frac{|< x, y >|^2}{\| y \|^2} + \frac{|< x, y >|^2}{\| y \|^2}$$

$$\therefore < x + \lambda y, x + \lambda y > = \| x \|^2 - \frac{|< x, y >|^2}{\| y \|^2}$$

$$\because < x + \lambda y, x + \lambda y > \geq 0 \quad [\text{since } < x, x > \geq 0]$$

$$\therefore \| x \|^2 - \frac{|< x, y >|^2}{\| y \|^2} \geq 0$$

$$\frac{|< x,y >|^2}{\| y \|^2} \leq \| x \|^2$$

$$|< x,y >|^2 \leq \| x \|^2 \| y \|^2$$

Therefor $|< x,y >| \leq \| x \| \| y \|$

Theorem3 : Every inner product space is norm space .

Proof : Let $(X, <.,.>)$ is an inner product space , we must prove that $(X, \| x \|)$ is normed space where $\| x \| = \sqrt{< x,x >}$.

$\because$ $(X, <.,.>)$ is an inner product space , then we get the following

1- $\sqrt{< x,x >} \geq 0$ $\quad$ $(since < x,x > \geq 0 \ \forall x \in X)$

$\because \| x \| = \sqrt{< x,x >}$

$\therefore \| x \| \geq 0 \quad \forall x \in X$

2- $\| x \| = 0 \leftrightarrow \sqrt{< x,x >} = 0 \leftrightarrow < x,x > = 0 \leftrightarrow x = 0$ $(since < x,x > = 0$ *iff* *x=0)*

$\therefore \| x \| = 0$ iff $x = 0$

3- $\| \alpha x \| = \sqrt{< \alpha x, \alpha x >} = \sqrt{\alpha\bar{\alpha} < x,x >} = \sqrt{|\alpha|^2 < x,x >} = |\alpha|\sqrt{< x,x >} = |\alpha| \| x \|$

4- $\because \| x \| = \sqrt{< x,x >}$

$\therefore \| x + y \| = \sqrt{< x + y, x + y >} \quad \rightarrow \| x + y \|^2 = < x + y, x + y >$

Now, $\| x + y \|^2 = < x + y, x + y > =$

Lemma: The inner product is a continuous function .

Proof :

Theorem 4: (The Parallelogram Low)

If x and y are two vectors in an inner product space , then

$$\| x + y \|^2 + \| x - y \|^2 = 2 \| x \|^2 + 2 \| y \|^2$$

Proof : Let x and y are two vectors in an inner product space , then

$\| x + y \|^2 + \| x - y \|^2 = < x + y, x + y > + < x - y, x - y >$

$= < x, x + y > + < y, x + y > + < x, x - y > - < y, x - y >$

$= < x, x > + < x, y > + < y, x > + < y, y > + < x, x > - < x, y > - < y, x > + < y, y >$

$= < x, x > + < y, y > + < x, x > + < y, y >$

$= 2 < x, x > + 2 < y, y >$

$= 2 \| x \|^2 + 2 \| y \|^2$

Theorem 5: (Jordan – Von Neumann)

A normed space is an inner product space iff the norm of normed space satisfies the parallelogram low .

(Without proof)

Note : Every inner product is norm , but the convers is not true (i.e not necessary every norm is inner product) .

The norm is inner product if the norm satisfies the parallelogram low .

<u>Example 3 :</u> Show that a normal space $C[a,b]$ under the norm define by

$\|f\| = \sup_{x\in[a,b]} \{|f(x)|\}$, $\forall f \in C[a,b]$ is not inner product space.

Solution :

$C[a,b]$ is not inner product space because it is not satisfies the parallelogram low

$$\| x+y \|^2 + \| x-y \|^2 = 2\| x \|^2 + 2\| y \|^2$$

Let $f:[a,b] \to R$ such that $f(x)=1$ and $g:[a,b] \to R$ such that $g(x) = \frac{x-a}{b-a}$

$\because f$ and g two continuous function

$\therefore f,g \in C[a,b]$

Now, $\|f\| = \sup_{x\in[a,b]} \{|f(x)|\} = \sup_{x\in[a,b]} \{|1|\} = 1$

$$\|g\| = \sup_{x\in[a,b]} \{|g(x)|\} = \sup_{x\in[a,b]} \left\{\left|\frac{x-a}{b-a}\right|\right\} = 1$$

$$\|f+g\| = \sup_{x\in[a,b]} \{|(f+g)(x)|\} = \sup_{x\in[a,b]} \{|f(x)+g(x)|\}$$

$$= \sup_{x\in[a,b]} \left\{\left|1+\frac{x-a}{b-a}\right|\right\} = 2$$

$$\|f-g\| = \sup_{x\in[a,b]} \{|(f-g)(x)|\} = \sup_{x\in[a,b]} \{|f(x)-g(x)|\}$$

$$= \sup_{x\in[a,b]} \left\{\left|1-\frac{x-a}{b-a}\right|\right\} = 1$$

$\therefore \|f+g\|^2 + \|f-g\|^2 = 2^2 + 1^2 = 5$

$2 \parallel x \parallel^2 + 2 \parallel y \parallel^2 = 2(1) + 2(1) = 4$

$\therefore \parallel x + y \parallel^2 + \parallel x - y \parallel^2 \neq 2 \parallel x \parallel^2 + 2 \parallel y \parallel^2$

$\therefore C[a, b]$ is not inner product space

<u>H.W</u> Show that a normal space $C[0,1]$ under the norm define by $\|f\| = \sup_{x \in [0,1]} \{|f(x)|\}$, $\forall f \in C[0,1]$ is not inner product space.