Chapter Four

Banach Space

Definition : (Cauchy sequence)

A sequence $\{x\}_{n=1}^{\infty}$ in a normed space X is Cauchy sequence if for any $\epsilon > 0$ there exists an integer number N such that $\|x_n - x_m\| < \epsilon$ for all $n, m > N$

Definition : (Complete Space)

The space X is complete if every Cauchy sequence in X is convergent to a point in X.

Note : R and C are complete space because every Cauchy sequence in R and in C are convergent

Definition : (Banach space)

The Banach space is a complete normed space.

Note :

Every finite dimensional normed space is a Banach space.

Example 1:

Show that a vector space R^n is a Banach space under the norm define by

$$\|x\| = [\sum_{i=1}^{n} | x_i|^2]^{\frac{1}{2}} \quad \forall \ x \in R^n.$$

Solution :

At first we prove that R^n is normed space [cheek].

Now we prove that R^n is a complete space :

Let $[x_m]_{m=1}^{\infty} = (x_1, x_2, \dots, x_m, \dots)$ be a Cauchy sequence in R^n.

$\therefore\ \forall x_m$ is n-tuple of real numbers

$\therefore$ we shall write $x_m = \left(x_1^m , x_2^m, \dots , x_n^m\right)$

Let $\epsilon > 0$

$\because\ [x_m]_{m=1}^{\infty}$ is a Cauchy sequence then , $\exists$ a positive integer $m_0 > 0$ such that

$\|x_m - x_L\| < \epsilon \quad \forall\, m, L > m_0$

$$\therefore \left[\sum_{i=1}^{n} \left|x_i^m - x_i^L\right|^2\right]^{\frac{1}{2}} < \epsilon \qquad \dots \ (1)$$

$\sum_{i=1}^{n} \left|x_i^m - x_i^L\right|^2 < \epsilon^2$

$$\left|x_i^m - x_i^L\right|^2 < \epsilon^2 \qquad \forall\, i = 1,2, \dots , n$$

$$\therefore \left|x_i^m - x_i^L\right| < \epsilon \qquad i = 1,2, \dots , n$$

This shows that the sequence $[x_i^m]_{m=1}^{\infty}$ is Cauchy sequence of real number for fixed

m but arbitrary i .

$\because R$ is complete

$\therefore$ Every Cauchy sequence in R is convergent to point in R .

$\therefore$ Cauchy sequence $[x_i^m]_{m=1}^{\infty}$ is convergent to point z_i in R .

$\lim_{m\to\infty} x_i^m = z_i \quad (i = 1,2, \dots , n)$

Now, we prove that $x_m \to z = (z_1, z_2, \dots , z_n)$

If $L \to \infty\ in\ \ (1)$

And for $m > m_0$ we have

$$\left[\sum_{i=1}^{n} |x_i^m - z_i|^2\right]^{\frac{1}{2}} < \epsilon$$

$\therefore \|x_m - z\| < \epsilon$

Thus , the Cauchy sequence $[x^m]_{m=1}^{\infty}$ convergence to $z \in R^n$

$\therefore$ R^n is complete .

$\therefore$ R^n is Banach space .

<u>Example 2:</u> **Show that a normed space $\boldsymbol{C[0,1]}$ where norm defined as follows :** $\|\boldsymbol{f}\| = \max\limits_{\boldsymbol{0 \leq x \leq 1}} \{|\boldsymbol{f(x)}|\} \quad \forall \boldsymbol{f} \in \boldsymbol{C[0,1]}$ **is Banach space.**

Solution :

$C[0,1]$ is normed space.

Now , we prove that $C[0,1]$ is complete,

let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $C[0,1]$

$\therefore$ $\forall \in > 0$, $\exists\, m_0 > 0$ such that

$\|f_m - f_n\| < \epsilon \quad \forall\, m, n > m_0$

$$\Longrightarrow \max_{0 \leq x \leq 1} \{|(f_m - f_n)(x)|\} < \epsilon$$

$$\Longrightarrow \max_{0 \leq x \leq 1} \{|f_m(x) - f_n(x)|\} < \epsilon$$

$\Longrightarrow |f_m(x) - f_n(x)| < \epsilon \quad \forall \quad 0 \leq x \leq 1$

But this is the condition for uniform convergence of the sequence of continuous real– valued function

$\therefore \{f_n\}_{n=1}^{\infty}$ must be convergent to a continuous function f in $C[0,1]$

i.e $f_n \to f \in C[0,1]$

$\therefore C[0,1]$ is complete space

$\therefore C[0,1]$ is Banach space.

Example 3: **Show that a linear space $\boldsymbol{C}(X)$ with norm defined as follows:**

$$\|\boldsymbol{f}\| = \sup_{\boldsymbol{x} \in \boldsymbol{X}} \{|\boldsymbol{f}(\boldsymbol{x})|\} \quad \forall \quad \boldsymbol{f} \in \boldsymbol{C}(X)$$ **is Banach space .**

Solution :

At first we prove that $\boldsymbol{C}(X)$ is normed space . H.W.

Now we prove that $\boldsymbol{C}(X)$ is complete :

let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $C[a,b]$

$\therefore\ \forall \in > 0$, $\exists\ m_0 > 0$ such that

$\|f_m - f_n\| < \epsilon \quad \forall\, m,n > m_0$

$$\Rightarrow \sup_{x \in X} \{|(f_m - f_n)(x)|\} < \epsilon$$

$$\Rightarrow \sup_{x \in X} \{|f_m(x) - f_n(x)|\} < \epsilon$$

$\Rightarrow |f_m(x) - f_n(x)| < \epsilon \quad \forall x \in X$

But this is the condition for uniform convergence of the sequence of continuous real– valued function

$\therefore \{f_n\}_{n=1}^{\infty}$ must be convergent to a continuous function f in $\boldsymbol{C}(X)$

i.e $f_n \to f \in \boldsymbol{C}(X)$

$\therefore C(X)$ is complete space

$\therefore C(X)$ is Banach space.

H.W.

1- Show that a vector space R is a Banach space under the norm define by :

$$\|x\| = |x| \quad \forall x \in R$$

2- Show that a vector space C is a Banach space under the norm define by :

$$\|z\| = |z| \quad \forall z \in C$$

3-Show that a vector space C^n is a Banach space under the norm define by :

$$\|z\| = [\sum_{i=1}^{n} |z_i|^2]^{\frac{1}{2}} \quad \forall z \in C^n.$$

4- Show that every Banach space is normed space , but the converse is not true .