Chapter Three

Normed Space

Definition : **(Normed space)**

Let X be a real or (complex) vector space and let $\| \ \|: X \to R$ be a function such that $\forall\, x, y \in X$ and $\alpha \in F$ satisfying the following :

1- $\|x\| \geq 0$

2- $\|x\| = 0$ iff $x = 0$

3- $\|\alpha x\| = |\alpha|\, \|x\|$

4- $\|x + y\| \leq \|x\| + \|y\|$

Then $(X, \| \ \|)$ is said to be a real or (complex) normed space .

Note :

1- If $(X_1, \| \ \ \|_1)$ and $(X_2, \| \ \ \|_2)$ are normed spaces, then the product $X_1 \times X_2$ is normed space .

2- Every finite dimensional subspace of a normed space is closed.

Example 1 : A vector space R^n is a normed space under the norm defines by:

$$\|x\| = [\sum_{i=1}^{n} |x_i|^2]^{\frac{1}{2}} \quad \forall x \in R^n, \text{ show that.}$$

Solution : $\forall x, y \in R^n$, $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$,

$$x_1, \ldots, x_n \text{ and } y_1, \ldots, y_n \in R$$

1- $\because |x_i| \geq 0 \quad \forall i = 1,2, \ldots, n$

$$\Rightarrow \sum_{i=1}^{n} |x_i|^2 \geq 0$$

$$\Rightarrow [\sum_{i=1}^{n} |x_i|^2]^{\frac{1}{2}} \geq 0$$

$$\Rightarrow \|x\| \geq 0 \quad \forall x \in C^n$$

2- let $\|x\| = [\sum_{i=1}^{n} |x_i|^2]^{\frac{1}{2}} = 0 \Leftrightarrow \sum_{i=1}^{n} |x_i|^2 = 0$

$$\Leftrightarrow |x_i|^2 = 0, \ \forall i = 1,2, \ldots, n$$

$$\Leftrightarrow |x_i| = 0 \ \forall i = 1,2, \ldots, n$$

$$\Leftrightarrow x_i = 0 \quad \forall i = 1,2, \ldots, n$$

$$\Leftrightarrow (x_1, x_2, \ldots, x_n) = (0,0, \ldots, 0) \quad \Leftrightarrow x = 0$$

3- $\|\alpha x\| = [\sum_{i=1}^{n} |\alpha x_i|^2]^{\frac{1}{2}} = [\sum_{i=1}^{n} |\alpha|^2 |x_i|^2]^{\frac{1}{2}} = [|\alpha|^2 \sum_{i=1}^{n} |x_i|^2]^{\frac{1}{2}}$

$$= [|\alpha|^2]^{\frac{1}{2}} . [\sum_{i=1}^{n} |x_i|^2]^{\frac{1}{2}} = |\alpha| . [\sum_{i=1}^{n} |x_i|^2]^{\frac{1}{2}} = |\alpha| . \|x\|$$

4- $x + y = (x_1 + y_1, \dots, x_n + y_n)$

$\|x + y\| = [\sum_{i=1}^{n} |x_i + y_i|^2]^{\frac{1}{2}}$ By Minkowski inequality

$$\leq [\sum_{i=1}^{n} |x_i|^2]^{\frac{1}{2}} + [\sum_{i=1}^{n} |y_i|^2]^{\frac{1}{2}}$$

$$\leq \|x\| + \|y\|$$

Example 2 :

Show that a vector space $C[a,b]$ is a normal space under the norm define :

by $\|f\| = \sup_{x \in [a,b]} \{|f(x)|\}$

Solution :

$1-\ \because |f(x)| \geq 0 \implies \sup_{x \in [a,b]} \{|f(x)|\} \geq 0$

$\therefore \|f\| \geq 0$

$2-\ \|f\| = 0 \iff \|f\| = \sup_{x \in [a,b]} \{|f(x)|\} = 0$

$\iff |f(x)| = 0, \forall\ x \in [a,b] \iff f(x) = 0, \forall\ x \in [a,b] \iff f = 0$

$3-\ \|\alpha . f\| = \sup_{x \in [a,b]} \{ |\alpha . f(x)| \} = \sup_{x \in [a,b]} \{ |\alpha| . |f(x)| \}$

$$= |\alpha| . \sup_{x \in [a,b]} \{|f(x)|\} = |\alpha| . \|f\|$$

$$4- \ \|f+g\| = \sup_{x\in[a,b]} \{|f(x)+g(x)|\} \leq \sup_{x\in[a,b]} \{|f(x)|+|g(x)|\}$$

$$\leq \sup_{x\in[a,b]} \{|f(x)|\} + \sup_{x\in[a,b]} \{|g(x)|\}$$

$$\leq \|f\|+\|g\|$$

H.W.

1- Show that a vector space R is a normed space under the norm define by :

$$\|x\| = |x| \quad \forall x \in R$$

2- Show that a vector space C is a normed space under the norm define by :

$$\|z\| = |z| \quad \forall z \in C$$

3- Show that a vector space C^n is a normed space under the norm define by :

$$\|z\| = [\sum_{i=1}^{n} |z_i|^2]^{\frac{1}{2}} \quad \forall z \in C^n$$

Note let $(X, \| \ \|)$ be a normed space and let $d: X \times X \rightarrow R$ defined by the $\boldsymbol{d(x,y) = \|x-y\|}$, then d is a metric induced by the norm.

Now, we prove that every normed space is matric space .

let $(X, \| \ \|)$ be a normed space and let $d: X \times X \rightarrow R$ defined by the

$$d(x,y) = \|x-y\|$$

1- $\because (X, \| \ \|)$ be a normed space $\rightarrow \| \ x \ \| \geq 0 \ \forall \ x \in X$

Let x, y $\in X \rightarrow$ x- y $\in X$ (since X is vector space)

$\therefore \|x-y\| \geq 0 \;\rightarrow\; d(x,y)$

2- let $d(x,y)=0 \;\leftrightarrow \|x-y\|=0 \leftrightarrow x-y=0$ (since $\|x\|=0$ iff $x=0$)

$\leftrightarrow x=y$

3 - $d(x,y)=\|x-y\|=\|-(y-x)\|=|-1|\|y-x\|$ (since $\|\alpha x\|=|\alpha|\,\|x\|$)

$=\|y-x\|=d(y,x)$

4- $d(x,y)=\|x-y\|=\|x-y-z+z\|=\|x-z)+(z-y)\|$

$\leq \|x-z\|+\|z-y\|$ (since $\|x+y\| \leq \|x\|+\|y\|$)

$\leq d(x,z)+d(z,y)$

Remark : $\forall\, z,w \in C$ then $\dfrac{|z+w|}{1+|z+w|} \leq \dfrac{|z|}{1+|z|}+\dfrac{|w|}{1+|w|}$

Example 5: Give any example to show that the metric space is not normed space .

Solution : let X be a set of all complex sequences $\{x_i\}$, and let $d: X \times X \rightarrow R$ is

define by $d(x,y)=\sum_{i=1}^{n} \frac{1}{2^i}\left(\frac{|x_i-y_i|}{1+|x_i-y_i|}\right)$

Now we proof that (X, d) is metric space : H.W.

This metric space is not be a normed space , because if there is norm such that

$d(x,y)=\|x-y\|$ then $d(\alpha x,\alpha y)=|\alpha| d(x,y)$ i.e $\|\alpha x-\alpha y\|=|\alpha|\|x-y\|$.

But $\|\alpha x-\alpha y\| \neq |\alpha|\|x-y\|$. H.W.