Example 5 : $(\boldsymbol{C}, \boldsymbol{d})$ **is metric space where** $\boldsymbol{d} : \boldsymbol{C} \times \boldsymbol{C} \to \boldsymbol{R}$ **is define by** $\boldsymbol{d(z,w) = |z - w|}$, $\forall\, \boldsymbol{z, w} \in \boldsymbol{C}$, **show that.**

Solution : let z, w and $h \in C$, 1, 2, 3 H.W

4- $d(z,w) = |z - w| = |z - w - h + h| = |(z - h) + (h - w)|$

$\because\ |z|^2 = z\bar{z}$

$|(z-h)+(h-w)|^2 = \big((z-h)+(h-w)\big)\big(\overline{(z-h)+(h-w)}\big)$

$= \big((z-h)+(h-w)\big)\big(\overline{(z-h)} + \overline{(h-w)}\big)$

$=(z-h)\,\overline{(z-h)} + (z-h)\overline{(h-w)} + (h-w)\overline{(z-h)} + (h-w)\overline{(h-w)}$

$= |z-h|^2 + (z-h)\overline{(h-w)} + \overline{(z-h)\overline{(h-w)}} + |h-w|^2$

$= |z-h|^2 + 2R\left|(z-h)\overline{(h-w)}\right| + |h-w|^2$ since [$z + \bar{z} = 2R|z|$]

$\leq |z-h|^2 + 2\left|(z-h)\overline{(h-w)}\right| + |h-w|^2$ since [$2R|z| \leq 2|z|$]

$\leq |z-h|^2 + 2|(z-h)|\left|\overline{(h-w)}\right| + |h-w|^2$ since [$|zw| = |z||w|$]

$\leq |z-h|^2 + 2|(z-h)||(h-w)| + |h-w|^2$ since [$|z| = |\bar{z}|$]

$\leq (|z-h| + |h-w|)^2$

$\therefore |(z-h)+(h-w)|^2 \leq (|z-h| + |h-w|)^2$

$|(z-h)+(h-w)| \leq |z-h| + |h-w|$

$\therefore d(z,w) \leq d(z,h) + d(h,w)$

Example 6 : **Show that** $\boldsymbol{C^n}$ **is a metric space where** $\boldsymbol{d} : \boldsymbol{C^n} \times \boldsymbol{C^n} \to \boldsymbol{R}$ **is define by** $\boldsymbol{d(z,w)} = [\ \sum_{i=1}^{n} |z_i - w_i|^2\ ]^{\frac{1}{2}}$, $\forall\, \boldsymbol{z, w} \in \boldsymbol{C^n}$.

Solution: $\forall\ z, w, h \in C^n \to z = (z_1, \dots, z_n), w = (w_1, \dots, w_n), and\ h = (h_1, \dots, h_n)$

$z_1, \dots, z_n, w_1, \dots, w_n, h_1, \dots, h_n \in C$.

1- $\because |z_i - w_i|^2 \geq 0\ \ \forall\ i = 1, 2, \dots, n$

$\therefore \sum_{i=1}^{n} |z_i - w_i|^2 \geq 0 \ \to [\ \sum_{i=1}^{n} |z_i - w_i|^2\ ]^{\frac{1}{2}} \geq 0$

$\therefore d(z,w) \geq 0$

2- $d(z,w) = 0 \leftrightarrow [\ \sum_{i=1}^{n}\ |z_i - w_i|^2\ ]^{\frac{1}{2}} = 0 \leftrightarrow \sum_{i=1}^{n}\ |z_i - w_i|^2 = 0$

$\leftrightarrow\ |z_i - w_i|^2 = 0\ \forall\ i = 1, 2, \dots, n\ \leftrightarrow z_i - w_i = 0\ \ \forall\ i = 1, 2, \dots, n$

$\leftrightarrow z_i = w_i\ \forall\ i = 1, 2, \dots, n \leftrightarrow (z_1, \dots, z_n) = (w_1, \dots, w_n) \leftrightarrow z = w$

3- $|z_i - w_i| = |-(w_i - z_i)| = |-1||(w_i - z_i)| = |w_i - z_i|\ \forall\ i = 1, 2, \dots, n$

$d(z,w) = [\ \sum_{i=1}^{n} |z_i - w_i|^2\ ]^{\frac{1}{2}} = [\ \sum_{i=1}^{n} |w_i - z_i|^2\ ]^{\frac{1}{2}} = d(w,z)$.

4- $d(z,w) = [\ \sum_{i=1}^{n} |z_i - w_i - h_i + h_i|^2\ ]^{\frac{1}{2}}$

$= [\ \sum_{i=1}^{n} |(z_i - h_i) + (h_i - w_i)|^2\ ]^{\frac{1}{2}}$ by Minkowski`s Inequality (Theorem 2)

$\leq [\ \sum_{i=1}^{n} |z_i - h_i|^2\ ]^{\frac{1}{2}} + [\ \sum_{i=1}^{n} |h_i - w_i|^2\ ]^{\frac{1}{2}}$

$\leq d(z,h) + d(h,w)$ $\therefore C^n$ is a metric space

H.W

1- Show that R^4 is a metric space where $d : R^4 \times R^4 \to R$ is define by

$$d(x,y) = [\ \sum_{i=1}^{4}(x_i - y_i)^2\ ]^{\frac{1}{2}}\ , \forall\ x,y\ \in R^4.$$

2- Show that C^3 is a metric space where $d : C^3 \times C^3 \to R$ is define by

$$d(z,w) = \left[\ \sum_{i=1}^{3}|z_i - w_i|^2\ \right]^{\frac{1}{2}}\ , \forall\ z,w\ \in C^3.$$

3- Show that R^2 is metric space where $d : R^2 \times R^2 \to R$ is define by

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

<u>Note :</u>

1- $C[a,b]$ denoted the set of all continuous real valued function on a closed interval $[a,b]$.

2- $B[a,b]$ denoted the set of all bounded real valued function on a closed interval $[a,b]$.

3- $R[a,b]$ denoted the class of all Riemann integrable function from $[a,b]$ in to R.

4- let $f: A \to B$ and $g: A \to B$ be any to function , we say that

$f = g$ if $f(x) = g(x)\quad \forall x \in A$.

<u>Example 7</u> : Show that $C[a,b]$ is metric space under $d : C[a,b] \times C[a,b] \to R$ define

by $$d(f,g) = \max_{t \in [a,b]} |f(t) - g(t)| \qquad \forall\ f,g\ \in\ C[a,b].$$

Solution : $C[a,b] = \{f | f: [a,b] \to R, f\ continuous\ real\ valued\ function\}$

Let $f,g\ \in\ C[a,b] \to f: [a,b] \to R\ and\ g: [a,b] \to R$

1- $\because |f(t) - g(t)| \geq 0\ ,\ \forall\ t \in [a,b]$

$$\therefore \max_{t \in [a,b]} |f(t) - g(t)| \geq 0 \quad \to \quad d(f,g) \geq 0$$

2- We must prove that $d(f,g) = 0 \leftrightarrow f = g$

$$d(f,g) = 0 \leftrightarrow \max_{t \in [a,b]} |f(t) - g(t)| = 0$$

$\leftrightarrow |f(t) - g(t)| = 0 \quad \forall\ t \in [a,b]$

$\leftrightarrow f(t) - g(t) = 0 \quad \forall\ t \in [a,b]$

$\leftrightarrow f(t) = g(t) \quad \forall\ t \in [a,b]$

$\leftrightarrow f = g$

3 and 4 H.W

$\therefore C[a,b]$ is metric space

Example 8:

Show that $\boldsymbol{R[0,1]}$ is pseudo metric space under $\boldsymbol{d : R[0,1] \times R[0,1] \rightarrow R}$ define

by $\boldsymbol{d(f,g) = \int_0^1 |f-g|(x)dx}$ $\quad \boldsymbol{\forall f,g \in R[0,1]}$.

Solution :

$R[0,1] = \{f \mid f\ has\ Riemann\ integrable\ function\ from\ [0,1]\ in\ to\ R\ \}$

Let $f,g \in R[0,1]$

$$d(f,g) = \int_0^1 |f-g|(x)dx = \int_0^1 |f(x)-g(x)|dx$$

1- $\because |f(x)-g(x)| \geq 0 \quad \forall x \in [0,1]$

$\therefore \int_0^1 |f(x)-g(x)|\, dx \geq 0$

$\therefore d(f,g) \geq 0$

2- We must prove that $(f = g \rightarrow d(f,g) = 0$ but $d(f,g) = 0 \nrightarrow f = g)$

let $f = g \rightarrow f(x) = g(x) \forall x \in [0,1]$

$\rightarrow f(x) - g(x) = 0 \ \forall x \in [0,1]$

$\rightarrow |f(x) - g(x)| = 0 \quad \forall x \in [0,1]$

$$\rightarrow \int_0^1 |f(x)-g(x)|\, dx = 0 \qquad \rightarrow d(f,g) = 0$$

Now let $d(f,g) = 0$ not necessary that $f = g$

for example : let $f \neq g$

$$f(x) = \begin{cases} 2 & x = 0 \\ 0 & 0 < x \leq 1 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1 & x = 0 \\ 0 & 0 < x \leq 1 \end{cases}$$

$$(f-g)(x) = f(x) - g(x) = \begin{cases} 1 & x = 0 \\ 0 & 0 < x \leq 1 \end{cases}$$

f and g has only one point of discontinuous at $x = 0$.

$f, g \in R[0,1]$

$d(f,g) = \int_0^1 |f-g|(x)dx = \int_0^1 |f(x)-g(x)|dx = \int_0^1 0\, dx = 0$.

$\therefore f \neq g$ but $d(f,g) = 0$

3 and 4 H .W

$\therefore R[0,1]$ is pseudo metric space .

H .W

1- Show that (R^2, d) is pseudo metric space under $d : R^2 \times R^2 \to R$, define by $d(x,y) = |x_1 - y_1|, \ \forall x,y \in R^2$.

2- Give example to show that R^2 is pseudo metric space .

3- Is $R[0,1]$ a metric space where $d : R[0,1] \times R[0,1] \to R$ define by $d(f,g) = \int_0^1 |f-g|(x)dx \qquad \forall f,g \in R[0,1]$. ? why ?

Definition : **(Bounded metric space)**

We say that (X,d) is a bounded metric space, if $\exists \ M > 0 \ \ such\ tha \ \ d(x,y) \le M$ $\forall x,y \in X$.

Note : we can define the bounded metric space by another way :
(X,d) is a bounded metric space if the diameter of X ($\delta(X) = sup \{d(x,y): x,y \in X \}$) is finite

Note : if (X,d) is a not bounded metric space we say that (X,d) is unbounded metric space

Example 9: let (X,d) be a metric space where $d : X \times X \to R$ is define by $\forall\, x,y \in X$

$$\boldsymbol{d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}}$$

Show that X is bounded metric space .

Solution: The diameter of X :

$\delta(X) = sup \{d(x,y): x,y \in X \} = 1 \to$ The diameter of X is finite

$\therefore (X,d)$ be a bounded metric space .

Note : d in example 9 is called discrete metric .

Example 10: **(R,d) is metric space where $d : R \times R \to R$ is define by $d(x,y) = |x-y|$, $\forall\, x,y \in R$. Is (R,d) bounded metric space ?**

Solution:

(R,d) unbounded metric space , since there is not $M > 0 \ such\ tha \ d(x,y) \le M$.