# محاضرات تحليل الدالي
## كورس الثاني
## قسم الرياضيات
## للمرحلة الرابعة

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# Functional Analysis

1- Metric space .
2- Vector space.
3- Normed space .
4- Banach space .
5- Inner product space .
6- Hilbert space .
7- Orthogonal and Orthonormal.
8- Linear Operator .
9- Hilbert adjoint Operator .

## References :

1- Kreyszig,Introductory Functional Analysis with Application, 1978 .
2- Dr.Ahmed Zain-Alabedeen,Elements of Functional Analysis,1986 .

# Chapter one
# Metric space

**Definition : (metric space )**

let $X$ be any non-empty set and let $d: X \times X \to R$ be any function. The pair $(X, d)$ is a metric space if $d$ satisfied the following :

1- $d(x,y) \geq 0$ , $\forall\, x,y \in X$.

2- $d(x,y) = 0 \leftrightarrow x = y$ , $\forall\, x,y \in X$.

3- $d(x,y) = d(y,x)$ , $\forall\, x,y \in X$.

4- $d(x,y) \leq d(x,z) + d(z,y)$ , $\forall\, x,y,z \in X$.

**Note** : 1- $d$ is called distance function or ( metric function ).

2- we can say that $X$ is metric space instead of $(X, d)$.

3- if the second condition dropped then we called $(X, d)$ is **semi- metric space** ( **pseudo metric space**).

i.e ( $x = y \to d(x,y) = 0$ but $d(x,y) = 0 \not\to x = y$ , $\forall\, x,y \in X$) .

**Example 1 : $(R, d)$ is metric space where $d : R \times R \to R$ is define by $d(x,y) = |x - y|$ , $\forall\, x,y \in R$, show that .**

**Solution :**

1- $\because |x - y| \geq 0, \ \forall\, x,y \in R.$

$\therefore\ d(x,y) \geq 0, \qquad \forall\, x,y \in R.$

2- $d(x,y) = \big|d(x,y) = |x - y|\big| = 0 \leftrightarrow x - y = 0 \leftrightarrow x = y\,, \forall\, x,y \in R.$

3- $d(x,y) = |x - y| = |-(y - x)| = |-1||y - x| = d(y,x)\,, \forall\, x,y \in R.$

4- $d(x,y) = |x - y| = |x - y - z + z| = |(x - z) + (z - y)|$

$\leq |(x - z)| + |(z - y)|$

$\leq d(x,z) + d(z,y)$ , $\forall\, x,y,z \in R.$

$\therefore (R, d)$ is metric space.

**Note :** $d$ in example 1 is called usual metric **.**

**Example 2 : Show that $R^2$ is a metric space where $d : R^2 \times R^2 \to R$ is define by $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$ , $\forall\, x,y \in R^2$.**

**Solution:**

Let $x,y \in R^2 \to x = (x_1, x_2)$ *and* $y = (y_1, y_2)$, $x_1, x_2, y_1, y_2 \in R$

1- $\because |x_1 - y_1| \geq 0$ *and* $|x_2 - y_2| \geq 0$ , $\forall\, x,y \in R^2.$

$\therefore |x_1 - y_1| + |x_2 - y_2| \geq 0\,, \forall\, x,y \in R^2$

$\therefore\ d(x,y) \geq 0, \forall\, x,y \in R^2$

2-Let $d(x,y) = 0 \leftrightarrow |x_1 - y_1| + |x_2 - y_2| = 0 \leftrightarrow |x_1 - y_1| = 0 \; and \; |x_2 - y_2| = 0$
$\leftrightarrow x_1 - y_1 = 0 \; and \; x_2 - y_2 = 0 \leftrightarrow x_1 = y_1 \; and \; x_2 = y_2$
$\leftrightarrow (x_1, x_2) = (y_1, y_2) \leftrightarrow x = y$

3- $d(x,y) = |x_1 - y_1| + |x_2 - y_2| = |-(y_1 - x_1)| + |-(y_2 - x_2)|$
$= |-1||y_1 - x_1| + |-1||y_2 - x_2| = |y_1 - x_1| + |y_2 - x_2| = d(y,x)$

4- $d(x,y) = |x_1 - y_1 - z_1 + z_1| + |x_2 - y_2 - z_2 + z_2|$
$= |(x_1 - z_1) + (z_1 - y_1)| + |(x_2 - z_2) + (z_2 - y_2)|$
$\leq |x_1 - z_1| + |z_1 - y_1| + |x_2 - z_2| + |z_2 - y_2|$
$\leq |x_1 - z_1| + |x_2 - z_2| + |z_1 - y_1| + |z_2 - y_2|$
$\leq d(x,z) + d(z,y)$

**Example 3 : Is $(\boldsymbol{R^2}, \boldsymbol{d})$ is a metric space under $\boldsymbol{d : R^2 \times R^2 \to R}$ ,**
**define by $\boldsymbol{d(x,y) = |x_1 - y_1|}$, $\forall \boldsymbol{x, y} \in \boldsymbol{R^2}$? Why ?**

**Solution :** let $x, y \in R^2 \to x = (x_1, x_2) \; and \; y = (y_1, y_2), \quad x_1, x_2, y_1, y_2 \in R$

1- $\because |x_1 - y_1| \geq 0, \; \forall \, x_1, y_1 \in R$
$\therefore \; d(x,y) \geq 0, \forall \, x, y \in R^2$

2- let $d(x,y) = 0 \to |x_1 - y_1| = 0 \to x_1 - y_1 = 0 \to x_1 = y_1 \nrightarrow x = y$, because :
that is not necessary $x_2 = y_2$ for example : (3,7) ≠ (3,1)
$\therefore \; d(x,y) = 0 \nrightarrow x = y \; , \; \forall \, x, y \in R^2$
$\therefore$ ( $R^2, d$) not metric space **.**

**H.W** Show that $R^2$ is metric space where $d : R^2 \times R^2 \to R$ is define by
$d(x,y) = \max \{|x_1 - y_1|, |x_2 - y_2|\}$

## Theorem 1 : ( Minkowski`s Inequality of real numbers)

If { $p_1, \; p_2, \; \dots , \; p_n$} and { $q_1, \; q_2, \; \dots , \; q_n$} be two sets of real numbers , then

$$[ \sum_{i=1}^{n} (p_i + q_i)^2 ]^{\frac{1}{2}} \leq [ \sum_{i=1}^{n} (p_i)^2 ]^{\frac{1}{2}} + [ \sum_{i=1}^{n} (q_i)^2 ]^{\frac{1}{2}}$$

## Theorem 2 : ( Minkowski`s Inequality of complex numbers)

{ $p_1, \; p_2, \; \dots , \; p_n$} and { $q_1, \; q_2, \; \dots , \; q_n$} be two sets of complex numbers , then

$$[ \sum_{i=1}^{n} |p_i + q_i|^2 ]^{\frac{1}{2}} \leq [ \sum_{i=1}^{n} |p_i|^2 ]^{\frac{1}{2}} + [ \sum_{i=1}^{n} |q_i|^2 ]^{\frac{1}{2}}$$

**Example 4** : **Show that $R^n$ is a metric space where $d : R^n \times R^n \to R$ is define by $d(x,y) = [\ \sum_{i=1}^{n}(x_i - y_i)^2\ ]^{\frac{1}{2}}$ , $\forall\ x, y\ \in R^n$.**

**Solution**:

$\forall\ x, y\ \in R^n \to x = (x_1, \dots, x_n)\ and\ \ y = (y_1, \dots, y_n),\ \ x_1, \dots, x_n, y_1, \dots, y_n \in R$

1, 2 , 3 H.W

4- $d(x,y) = [\ \sum_{i=1}^{n}(x_i - y_i - z_i + z_i)^2\ ]^{\frac{1}{2}}$

$= [\ \sum_{i=1}^{n}((x_i - z_i) + (z_i - y_i))^2\ ]^{\frac{1}{2}}$ by Minkowski`s Inequality (Theorem 1)

$\leq [\ \sum_{i=1}^{n}((x_i - z_i))^2\ ]^{\frac{1}{2}} + [\ \sum_{i=1}^{n}(z_i - y_i)^2\ ]^{\frac{1}{2}}$

$\leq\ d(x,z) + d(z,y)$