6- Solving Systems of Differential Equations Using Laplace Transform.

In systems of differential equations we have $y(t)$ (or y) then Laplace transform of derivations becomes: let $L\{y(t)\} = Y(s)$

1- $\boldsymbol{L\{\dot{y}\} = sY(s) - y(0)}$

2- $\boldsymbol{L\{y''\} = s^2Y(s) - sy(0) - y'(0)}$

3- $\boldsymbol{L\{y'''(t)\} = s^3Y(s) - s^2y(0) - sy'(0) - y''(0)}$

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$$\boldsymbol{L\{y^n(t)\} = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - s^{n-3}y''(0) - \cdots - y^{n-1}(0)}$$

<u>Note</u> to solve linear **differential equation** (initial value problems I.V.P) using Laplace Transforms, there are only 4 basic steps :

1- Take Laplace Transforms of both sides of an equation**.**

2- using Laplace transform of derivations.

3- Simplify algebraically the result to find $Y(s)$.

4- Take L^{-1} of both sides , then we get $y(t)$ which is the solution of given equation.

<u>Example 1</u> : Using Laplace Transform to solve this deferential equations :

$$\boldsymbol{y' + 3y = e^{2t}} \qquad \boldsymbol{y(0) = -1}$$

Solution

ناخذ تحويل لابلاس لطرفي المعادلة :

$$L\{y'\} + 3L\{y\} = L\{e^{2t}\}$$

نعوض كل حد بما يساويه حسب قوانين تحويلات لابلاس للمشتقات وحسب الجدول :

$$sY(s) - y(0) + 3Y(s) = \frac{1}{s-2}$$

وبالاستفادة من الشروط الابتدائية المعطاة وتعويضها في المعادلة نحصل على :

$$(s+3)Y(s) = \frac{1}{s-2} - 1$$

$$(s+3)Y(s) = \frac{1-s+2}{s-2}$$

$$(s+3)Y(s) = \frac{-s+3}{(s-2)}$$

$$Y(s) = \frac{3-s}{(s-2)(s+3)}$$

وباستخدام طريقة تجزئة كسور سوف نجزء الكسر سابق كما يلي :

$$\frac{3-s}{(s+2)(s+3)} = \frac{A}{(s-2)} + \frac{B}{(s+3)}$$

$$\left.\frac{3-s}{(s+3)}\right|_{s=2} = \frac{1}{5} \qquad \therefore\ A = \frac{1}{5}$$

$$\left.\frac{3-s}{(s-2)}\right|_{s=-3} = \frac{-6}{5} \qquad \therefore\ B = \frac{-6}{5}$$

$$\therefore Y(s) = \frac{\frac{1}{5}}{(s-2)} + \frac{\frac{-6}{5}}{(s+3)}$$

ناخذ الان تحويل لابلاس العكسي لطرفين وبذلك نحصل على حل المعادلة $y(t)$:

$$L^{-1}[Y(s)] = L^{-1}\left[\frac{\frac{1}{5}}{(s-2)}\right] + L^{-1}\left[\frac{\frac{-6}{5}}{(s+3)}\right]$$

$$y(t) = \frac{1}{5}L^{-1}\left[\frac{1}{(s-2)}\right] - \frac{6}{5}L^{-1}\left[\frac{1}{(s+3)}\right]$$

$$y(t) = \frac{1}{5}e^{2t} - \frac{6}{5}e^{-3t}$$

<u>Example 2</u> : Using Laplace Transform to solve this I.V.P :

$$\boldsymbol{y'' + 2y' + y = 4\,, \qquad y(0) = 4\ , y'(0) = 2}$$

Solution :

$$L\{y''\} - 2L\{y'\} + L\{y\} = L\{4\}$$

$$s^2Y(s) - sy(0) - y'(0) - 2[sY(s) - y(0)] + Y(s) = \frac{4}{s} \quad \ldots\ldots\ldots \quad (1)$$

$$s^2Y(s) - 4s - 2 - 2sY(s) + 8 + Y(s) = \frac{4}{s}$$

$$Y(s)(s^2 - 2s + 1) = 4s - 6 + \frac{4}{s}$$

$$Y(s) = \frac{4s^2 - 6s + 4}{s(s^2 - 2s + 1)} = \frac{4s^2 - 6s + 4}{s(s-1)^2}$$

$$\frac{4s^2 - 6s + 4}{s(s-1)^2} = \frac{A}{s} + \frac{B}{(s-1)} + \frac{C}{(s-1)^2}$$

$$= \frac{A(s-1)^2 + B(s(s-1)^2) + Cs}{s(s-1)^2}$$

$$\therefore\ 4s^2 - 6s + 4 = As^2 - 2As + A + Bs^2 + Bs + Cs$$

$$= (A+B)s^2 + (-2A - B + C)S + A$$

$$\therefore\ A + B = 4$$

$-2A - B + C = -6$

$\therefore \quad A = 4\,,\ B = 0\ \ , C = 2$

$$\therefore Y(s) = \frac{4}{s} + \frac{2}{(s-1)^2}$$

$$L^{-1}[Y(s)] = L^{1-}\left[\frac{4}{5}\right] + L^{-1}\left[\frac{2}{(s-1)^2}\right]$$

$$\therefore y(t) = 4 + 2te^{t}$$

<u>Example 3</u> : Using Laplace Transform to solve this deferential equation :

$$\boldsymbol{y'' + y = \sin 2t} \ , \qquad \boldsymbol{y(0) = 0} \ \ , \boldsymbol{y'(0) = 1}$$

Solution :

$L\{y''\} + L\{y\} = L\{\sin 2t\}$

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{2}{s^2+4}$$

$$s^2Y(s) - 0 - 1 + Y(s) = \frac{2}{s^2+4}$$

$$(s^2+1)Y(s) = \frac{s^2+6}{s^2+4}$$

$$Y(s) = \frac{s^2 + 6}{(s^2 + 4)(s^2 + 1)}$$

$$\frac{s^2 + 6}{(s^2 + 4)(s^2 + 1)} = \frac{As + B}{(s^2 + 4)} + \frac{(s + D)}{(s^2 + 1)}$$

$$= \frac{As + B(s^2 + 1)(s + D)(s^2 + 4)}{(s^2 + 4)(s^2 + 1)}$$

$$= \frac{(As^3 + As + Bs^2 - B) + (Cs^3 + 4Cs + Ds^2 + 4D)}{(s^2 + 4)(s^2 + 1)}$$

$A + C = 0 \qquad (1)$

$B + D = 1 \qquad (2)$

$A + 4C = 0 \qquad (3)$

$B + 4D = 6 \qquad (4)$

$$A = 0 \quad , B = \frac{-2}{3} \quad , C = 0 \quad , D = \frac{5}{3}$$

$$Y(s) = \frac{0 + \left(\frac{-2}{3}\right)}{(s^2 + 4)} + \frac{0 + \left(\frac{5}{3}\right)}{(s^2 + 1)}$$

$$L^{-1}[Y(s)] = L^{-1}\left[\frac{\left(\frac{-2}{3}\right)}{(s^2 + 4)}\right] + L^{-1}\left[\frac{\frac{5}{3}}{(s^2 + 1)}\right]$$

$$y(t) = \frac{-2}{3} \cdot \frac{1}{2} L^{-1}\left[\frac{2}{(s^2+4)}\right] + \frac{5}{4} L^{-1}\left[\frac{1}{(s^2+1)}\right]$$

$$y(t) = -\frac{1}{3}\sin 2t + \frac{5}{3}\sin t$$

<u>Example 4</u> : Using Laplace Transform to solve this deferential equation :

$$\boldsymbol{y''' - 3y'' + 3y' - y = t^2 e^t \qquad y(0) = 1 \ \ , y'(0) = 0 \ \ , y''(0) = 2}$$

Solution :

$$L\{y'''\} - 3L\{y''\} + 3L\{y'\} - L\{y\} = L\,\{t^2 e^t\}$$

$$[s^3 Y(s) - s^2 y(0) - sy'(0) - y''(0)] - [3s^2 Y(s) - 3sy(0) - sy'(0)] + [3sY(s) - 3y(0)] - Y(s) = \frac{2}{(s-1)^3}$$

$$s^3 Y(s) - s^2 - 0 - 2 - 3s^2 Y(s) + 3s - 0 + 3sY(s) - 3 - Y(s) = \frac{2}{(s-1)^3}$$

$$Y(s)[s^3 - 3s^2 + 3s - 1] = \frac{2}{(s-1)^3} + (s^2 - 3s + 5)$$

$$Y(s)(s-1)^3 = \frac{2}{(s-1)^3} + (s^2 - 3s + 5)$$

$$Y(s) = \frac{2}{(s-1)^6} + \frac{(s^2 - 3s + 5)}{(s-1)^3}$$

$$\frac{(s^2-3s+5)}{(s-1)^3}=\frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{C}{(s-1)^3}$$

$$A=1, \quad B=-1, \quad C=3 \qquad (H.W.)$$

$$\therefore Y(s)=\frac{2}{(s-1)^6}+\frac{1}{s-1}-\frac{1}{(s-1)^2}+\frac{3}{(s-1)^3}$$

$$L^{-1}[Y(s)]=L^{-1}\left[\frac{2}{(s-1)^6}\right]+L^{-1}\left[\frac{1}{s-1}\right]-L^{-1}\left[\frac{1}{(s-1)^2}\right]+L^{-1}\left[\frac{3}{(s-1)^3}\right]$$

$$\therefore \boldsymbol{y(t)=\frac{2}{5!}t^5e^t+e^t-te^t+\frac{3}{2}t^2e^t}$$

H.W. **Using Laplace Transform to solve the following deferential equations :**

1) $y'+3y=e^{2t}$ $\qquad y(0)=-1$

2) $3y''-4y'+4y=0$ $\qquad y(0)=1 \ , y'(0)=1$

3) $y''-2y'+2y=e^{-t}$ $\qquad y(0)=0 \ , y'(0)=1$

4) $y''+4y=8\sin t$ $\qquad y(0)=0 \ , y'(0)=2$

5) $y''-6y'+9y=t^2e^{3t}$ $\qquad y(0)=2 \ , y'(0)=6$

<u>Example 4 :</u> Using Laplace Transform to solve this system of deferential equations

$$2x' + 3x + y = 0$$

$$2y' + x + 3y = 0 \qquad \text{where} \quad x(0) = 2 \ \ and \ \ y(0) = 0$$

Solution :

$$2x' + 3x + y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$2L\{x'\} + 3L\{x\} + L\{y\} = L\{0\}$$

$$2sX(s) - 2x(0) + 3X(s) + Y(s) = 0$$

$$(2s+3)X(s) - 4 + Y(s) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1^{`})$$

$$2y' + x + 3y = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$2L\{y'\} + L\{x\} + 3L\{y\} = L\{0\}$$

$$2sY(s) - 2y(0) + X(s) + 3Y(s) = 0$$

$$(2s+3)Y(s) - 0 + X(s) = 0$$

$$Y(s) = \frac{-X(s)}{(2s+3)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2^{`})$$

By (1`) and (2`) we get :

$$(2s+3)X(s) - 4 - \frac{X(s)}{(2s+3)} = 0$$

$$(2s+3)^2X(s) - 4(2s+3) - X(s) = 0$$

$$((2s+3)^2 - 1)X(s) = 8s + 12$$

$$X(s) = \frac{8s+12}{(2s+3)^2 \ -1} = \frac{8s+12}{4s^2+12s+9-1} = \frac{8s+12}{4s^2+12s+8} = \frac{4(2s+3)}{4(s^2+3s+2)}$$

$$\therefore X(s) = \frac{(2s+3)}{(s^2+3s+2)} = \frac{(2s+3)}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

A=1 and B= 1

$$\therefore x(t) = L^{-1}[x(s)] = L^{-1}\left[\frac{1}{s+1}\right] + L^{-1}\left[\frac{1}{s+2}\right]$$

$$\therefore \boldsymbol{x(t) = e^{-t} + e^{-2t}}$$

$$\therefore x'(t) = -e^{-t} - 2e^{-2t}$$

by (1) we get : $2(-e^{-t} - 2e^{-2t}) + 3(e^{-t} + e^{-2t}) + y = 0$

$$-2e^{-t} - 4e^{-2t} + 3e^{-t} + 3e^{-2t} + y = 0$$

$$e^{-t} - e^{-2t} + y = 0$$

$$\boldsymbol{y\,(t) = -e^{-t} + e^{-2t}}$$

H.W. Using Laplace Transform to solve the following system of deferential equations:

1) $\boldsymbol{y' = -z}$

$\boldsymbol{z' = y}$ $\quad\quad$ $\boldsymbol{y(0) = 1}$ $\quad$ $\boldsymbol{and \ \ z(0) = 0}$

2) $\boldsymbol{x' - 2x - 3y = 0}$

$\boldsymbol{y' - 2x - y = 0}$ $\quad\quad$ **where** $\boldsymbol{x(0) = 2 \ \ and \ \ y(0) = 3}$