5- Invers Laplace Transform.

<u>Definition :</u>

Let $f(t)$, $t \geq 0$ be any function and let $L\{f(t)\} = F(s)$, the invers Laplace Transform of $f(t)$ is define by $L^{-1}[F(s)] = f(t)$

1- Find the invers Laplace Transform by using this table directly :

	$F(s)$	$f(t) = L^{-1}[F(s)]$
1	$\frac{1}{s}$	1
2	$\frac{1}{s^2}$	t
3	$\frac{n!}{s^{n+1}}$	t^n
4	$\frac{1}{s-a}$	e^{at}
5	$\frac{a}{s^2+a^2}$	$sinat$
6	$\frac{s}{s^2+a^2}$	$cosat$
7	$\frac{a}{s^2-a^2}$	$\sinh at$
8	$\frac{s}{s^2-a^2}$	$cosh\ at$

<u>Example 1</u> **:** find $L^{-1}\left[\frac{1}{s-7}\right]$

Solution :

$$L^{-1}\left[\frac{1}{s-7}\right] = e^{7t}$$

Example 2 : find $L^{-1}\left[\frac{3}{s^4}\right]$

Solution :

$$L^{-1}\left[\frac{3}{s^4}\right]=\frac{1}{2}L^{-1}\left[\frac{3.2.1}{s^4}\right]=\frac{1}{2}L^{-1}\left[\frac{3!}{s^4}\right]=\frac{1}{2}\ t^3$$

Example 3 : find $L^{-1}\left[\frac{1}{s^2+4}\right]$

Solution :

$$L^{-1}\left[\frac{1}{s^2+4}\right]=\frac{1}{2}\ L^{-1}\left[\frac{2}{s^2+2^2}\right]=\frac{1}{2}\sin 2t$$

Example 4 : **find** $L^{-1}\left[\frac{3}{s^2+25}\right]$

Solution :

$$L^{-1}\left[\frac{3}{s^2+25}\right]=\ 3L^{-1}\left[\frac{1}{s^2+5^2}\right]=\ \frac{3}{5}L^{-1}\left[\frac{5}{s^2+5^2}\right]=\frac{3}{5}\sin 5t$$

Example 5 : **find** $L^{-1}\left[\frac{4s}{s^2+16}\right]$

Solution :

$$L^{-1}\left[\frac{4s}{s^2+16}\right]=4L^{-1}\left[\frac{s}{s^2+16}\right]=\ 4\cos 4t$$

Example 6 : **find** $L^{-1}\left[\frac{1}{(s+3)^4}\right]$

Solution :

$$L^{-1}\left[\frac{1}{(s+3)^4}\right]= \frac{6}{6}L^{-1}\left[\frac{1}{(s+3)^4}\right]$$

$$= \frac{1}{6}L^{-1}\left[\frac{6}{(s+3)^4}\right]=\frac{1}{6}L^{-1}\left[\frac{3!}{(s+3)^4}\right] = \frac{1}{6}\,e^{-3t}\,t^3$$

Example 7 : **find** $L^{-1}\left[\frac{3s-1}{s^2+2s+5}\right]$

Solution :

$$\frac{3s-1}{s^2+2s+5}=\frac{3s-1}{s^2+2s+5-1+1}=\frac{3s-1}{(s^2+2s+1)+4}=\frac{3s-1}{(s+1)^2+4}$$

$$=\frac{3s+3-3-1}{(s+1)^2+2^2}=\frac{(3s+3)-4}{(s+1)^2+2^2}=\frac{3(s+1)-(2)^2}{(s+1)^2+2^2}$$

$$=\frac{3(s+1)}{(s+1)^2+2^2}-\frac{2^2}{(s+1)^2+2^2}$$

$$\therefore L^{-1}\left[\frac{3s-1}{s^2+2s+5}\right]=L^{-1}\left[\frac{3(s+1)}{(s+1)^2+2^2}-\frac{2^2}{(s+1)^2+2^2}\right]$$

$$=L^{-1}\left[\frac{3(s+1)}{(s+1)^2+2^2}\right]-L^{-1}\left[\frac{2^2}{(s+1)^2+2^2}\right]$$

$$= 3L^{-1}\left[\frac{s+1}{(s+1)^2+2^2}\right] - 2L^{-1}\left[\frac{2^2}{(s+1)^2+2^2}\right]$$

$$= 3e^{-t}\, cos2t - 2e^{-t}\sin 2t \quad = e^{-t}(3\, cos2t - 2\sin 2t)$$

H.W

1- Compute $L^{-1}\left[\frac{2}{(s+5)^4}\right]$.

2- Compute $L^{-1}\left[\frac{s+1}{s^2-4}\right]$.

3- Evaluate $L^{-1}\left[\frac{8}{s^4-4s^2}\right]$.

4- Find $L^{-1}\left[\frac{s+1}{s^2+4}\right]$.

5- Let $F(s) = \frac{2-\sqrt{3s^2}}{4s^2-1}$ find $f(t)$.

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2- The Laplace Transform and the method of partial fraction

Notes :

$$1- \quad \frac{?}{(as+b)^n} = \frac{A_1}{(as+b)} + \frac{A_2}{(as+b)^2} + \frac{A_3}{(as+b)^3} + \cdots + \frac{A_n}{(as+b)^n}$$

$$2- \quad \frac{?}{as^2+bs+c} = \frac{A\,s+B}{as^2+bs+c}$$

$$3\text{-} \quad \frac{?}{(as^2+bs+c)^n} = \frac{A_1s+B_1}{as^2+bs+c} + \frac{A_2s+B_2}{(as^2+bs+c)^2} + \cdots + \frac{A_ns+B_n}{(as^2+bs+c)^n}$$

Example 8 : **find** $\boldsymbol{L^{-1}\left[\frac{s+1}{s^2(s-1)}\right]}$

Solution :

$$\frac{s+1}{s^2(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1}$$

$$\frac{s+1}{s^2(s-1)} = \frac{As(s-1)+B(s-1)+Cs^2}{s^2(s-1)}$$

$$\therefore s+1 = As(s-1)+B(s-1)+Cs^2$$

$$s+1 = As^2 - A + Bs - B + Cs^2$$

$$s+1 = (A+C)s^2 + (-A-B) + Bs$$

$$\therefore A+C = 0 \quad \rightarrow \quad A = -C$$

$$-A-B = 1$$

$B = 1$

$\therefore -A - 1 = 1 \;\rightarrow A = -2$

$C = 2$

$$\therefore L^{-1}\left[\frac{s+1}{s^2(s-1)}\right] = L^{-1}\left[\frac{-2}{s} + \frac{1}{s^2} + \frac{2}{s-1}\right]$$

$$= -2L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s^2}\right] + 2L^{-1}\left[\frac{1}{s-1}\right]$$

$$= -2(1) + t + 2e^t = -2 + t + 2e^t$$

<u>Example 9</u> : find $f(t)$ such that $F(s) = \frac{5s^2-7s+17}{s^3-s^2+45+4}$

Solution :

$\because f(t) = L^{-1}[F(s)]$

$$\therefore\; f(t) = L^{-1}\left[\frac{5s^2-7s+17}{s^3-s^2+45+4}\right]$$

$$\frac{5s^2-7s+17}{s^3-s^2+45+4} = \frac{5s^2-7s+17}{s^2(s-1)+4(s-1)} = \frac{5s^2-7s+17}{(s-1)(s^2+4)}$$

$$= \frac{A}{(s-1)} + \frac{Bs+C}{s^2+4} = \frac{As^2+4A+Bs^2-Bs+Cs-C}{(s-1)(s^2+4)}$$

$$= \frac{(A+B)\,s^2 + (C-B)\,s + 4A - C}{(s-1)\,(s^2+4)}$$

$A + B = 5 \rightarrow \;\; A = -B + 5$

$C - B = -7 \rightarrow \ C = B - 7$

$4A - C = 17 \quad \rightarrow \ \ 4(-B + 5) - (B - 7) = 17$

$\rightarrow -4B + 20 - B + 7 = 17 \rightarrow -5B = -10 \ \rightarrow B = 2$

$A = -2 + 5 \ \ \rightarrow \ A = 3$

$C = 2 - 7 \ \rightarrow C = -5$

$$\therefore L^{-1}\left[\frac{5s^2 - 7s + 17}{s^3 - 2^2 + 4s - 4}\right] = L^{-1}\left[\frac{3 + 2s - 5}{(s-1)(s^2+4)}\right]$$

$$= L^{-1}\left[\frac{3}{(s-1)}\right] + L^{-1}\left[\frac{2s-5}{s^2+4}\right] = 3L^{-1}\left[\frac{1}{s-1}\right] + L^{-1}\left[\frac{2s}{s^2+4} + \frac{-5}{s^2+4}\right]$$

$$= 3L^{-1}\left[\frac{1}{s-1}\right] + 2L^{-1}\left[\frac{s}{s^2+4}\right] - \frac{5}{2}L^{-1}\left[\frac{2}{s^2+4}\right]$$

$$= 3e^t t + 2\cos 2t - \frac{5}{2}\sin 2t$$

H.W.

1- Show that $L^{-1}\left[\frac{s^2+2}{s(s+1)(s+2)}\right]$=1+3($e^{-2t}$ – e^{-t}).

2- Compute $L^{-1}\left[\frac{1}{s(s+1)}\right]$.

3- Evaluate $L^{-1}\left[\frac{2s^2}{(s^2+1)(s-1)^2}\right]$

4- Let $F(s) = \frac{s+1}{s^2(s-1)}$ find $f(t)$.

5- Compute $L^{-1}\left[\frac{1}{(s-2)(s+3)}\right]$.

6- Find $L^{-1}\left[\frac{s+1}{s^2+6s+13}\right]$.

7- Compute $L^{-1}\left[\frac{s^2}{(s+3)^4}\right]$