3- Properties of Laplace Transform.

1- Let $\boldsymbol{f(t)}$ and $\boldsymbol{g(t)}$ be two functions have Laplace Transform $\boldsymbol{F(s)}$ and $\boldsymbol{G(s)}$ respectively, and let $\boldsymbol{c_1, c_2}$ be any two scalar, then

$$\boldsymbol{L\{c_1 f(t) + c_2 g(t)\} = c_1 L\{f(t)\} + c_2 L\{g(t)\} = c_1 F(s) + c_2 G(s)}$$

Proof : $L\{c_1 f(t) + c_2 g(t)\} = \int_0^{\infty} e^{-st}(c_1 f(t) + c_2 g(t)dt\ )$

$$= \int_0^{\infty} e^{-st}\big(c_1 f(t)\big)dt + \int_0^{\infty} e^{-st}\big(c_2\ g(t)\big)dt$$

$$= c_1 \int_0^{\infty} e^{-st} f(t)dt + c_2 \int_0^{\infty} g(t)e^{-st}dt$$

$$= c_1 L\{f(t)\} + c_2 L\{g(t)\} = c_1 F(s) + c_2 G(s)$$

Example :**let** $\boldsymbol{f(t) = t^3 + 5t - 2}$ **, find $\boldsymbol{F(s)}$.**

Solution: $L\{f(t)\} = L\{t^3 + 5t - 2\} = L\{t^3\} + 5L\{t\} - 2L\{1\}$

$$= \frac{3!}{s^4} + 5\left(\frac{1}{s^2}\right) - 2\left(\frac{1}{s}\right) = \frac{3.2.1}{s^4} + \frac{5}{s^2} - \frac{2}{s} = \frac{6}{s^4} + \frac{5}{s^2} - \frac{2}{s}$$

H.W

Find Laplace Transform of the following function :

1- $f(t) = 3t^2 + 2$

2- $f(t) = 2t^5 + sin3t + e^{-9t}$

3- $f(t) = 7\ e^{-3t} - \frac{cos5t}{3}$

2- $\boldsymbol{L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)}$ **where** $\boldsymbol{F(s) = L\{f(t)\}}$**.**

Proof : $F(s) = L\{f(t)\} = \int_0^\infty f(t) e^{-st} dt$

$$\frac{d}{ds}F(s) = \frac{d}{ds}\left(\int_0^\infty e^{-st} f(t)dt\right) = \left(\int_0^\infty \frac{d}{ds}(e^{-st}) f(t)dt\right) = \int_0^\infty (-t\, e^{-st}) f(t)dt$$

$$= -\int_0^\infty e^{-st}\big(tf(t)\big)dt = -L\{tf(t)\}$$

$\therefore L\{tf(t)\} = \frac{-d}{ds}\ F(s)$ if *n=1*

$$\frac{d^2}{ds^2}\ F(s) = \frac{d^2}{ds^2}\int_0^\infty e^{-st} f(t)dt = \frac{d}{ds}\left(\frac{d}{ds}\int_0^\infty e^{-st} f(t)dt\right) = \frac{d}{ds}\int_0^\infty e^{-st}(-t\, f(t))dt$$

$$= \int_0^\infty -t\, e^{-st}\big(-t\, f(t)\big)dt = \int_0^\infty e^{-st}\big(t^2\ f(t)\big)dt = L\{t^2 f(t)\}$$

$\therefore L\{t^2 f(t)\} = \frac{d^2}{ds^2}\ F(s)$ if *n=2*

$$\frac{d^3}{ds^3} = \frac{d^3}{ds^3}\left(\int_0^\infty f(t)\, e^{-st} dt\right) = \frac{d}{ds}\left(\frac{d^2}{ds^2}\int_0^\infty f(t)\, e^{-st} dt\right) = \frac{d}{ds}\int_0^\infty e^{-st}\,\big(t^2 f(t)\big)dt$$

$$= \int_0^\infty (-t\, e^{-st})\big(t^2 f(t)\big)dt = -\int_0^\infty e^{-st}\big(t^3 f(t)\big)dt = -L\{t^3 f(t)\}$$

$\therefore L\{t^3 f(t)\} = -\frac{d^3}{ds^3}\ F(s)$ if *n=3*

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$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\ F(s)$$

Example : Compute $\boldsymbol{L\{t^2 sin2t\}}$.

Solution : by $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$

$f(t) = \sin 2t \quad , t^n = t^2 \Longrightarrow n = 2$

$$L\{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2}\, F(s)$$

$$F(s) = L\{f(t)\} = L\{\sin 2t\} = \frac{2}{s^2+4}$$

$$L\{\mathrm{t}^2 \sin 2t\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{2}{s^2+4}\right) = \frac{d}{ds}\left(\frac{d}{ds}\left(\frac{2}{s^2+4}\right)\right)$$

$$= \frac{d}{ds}\left(\frac{(s^2+4)(0) - 2(2s)}{(s^2+4)^2}\right) = \frac{d}{ds}\left(\frac{-4s}{(s^2+4)^2}\right)$$

$$= \frac{(s^2+4)^2(-4) - (-4s)(2(s^2+4)(2s))}{(s^2+4)^4} = \frac{-4(s^2+4)^2 \ + 16s^2(s^2+4)}{(s^2+4)^4}$$

$$= \frac{(s^2+4)\,[-4(s^2+4)+16s^2]}{(s^2+4)^4} = \frac{-4(s^2+4)+16s^2}{(s^2+4)^3}$$

H.W

1- find $L\{t^3 \sin\ 5t\}$

2- find $L\{6t^2 cos\ 3t\}$

3- find $L\{-t^2 e^{3t}\}$

4- Show that $L\{\ t^3\ f(t)\} = \ - \frac{d^3}{ds^3}\ F(s)$

5- Show that $L\{\ t^2\ f(t)\} = \ \frac{d^2}{ds^2}\ F(s)$

6- Show that $L\{\ t^5\ f(t)\} = \ - \frac{d^5}{ds^5}\ F(s)$

3- Laplace transform of derivation

1- let $f(t), t \geq 0$ be differentiable function then

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

Proof :

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\, dt$$

$$let\ u = e^{-st} \rightarrow du = -se^{-st}\, dt$$

$$dv = f'(t)dt \quad \rightarrow v = f(t)$$

$$\therefore L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)dt = \left. e^{-st} f(t)\right]_0^\infty - (-s)\int_0^\infty e^{-st} f(t)dt$$

$$= \lim_{A\to\infty} e^{-st} f(t)\Big]_0^A + s\int_0^\infty e^{-st} f(t)dt = \left(\lim_{A\to\infty} e^{-A} f(A) - f(0)\right) + sL\{f(t)\}$$

$$= 0 - f(0) + sL\{f(t)\} = sL\{f(t) - f(0)\}$$

Example : find $L\{\cos t\}$ by using the first derivation .

Solution :

$L\{f'(t)\} = sL\{f(t)\} - f(0)$ ....(1)

$f(t) = \cos t \rightarrow \boldsymbol{f(0) = \cos 0 = 1}$ ... ... (2)

$f'(t) = -\sin t$

$L\{f'(t)\} = L\{-\sin t\}$

$$\boldsymbol{L\{f'(t)\} = \frac{-1}{s^2+1} \ldots\ldots\ldots\ldots (3)}$$

Now by (1) , (2) and (3) we get :

$$\frac{-1}{s^2+1} = sL\{\cos t\} - 1$$

$$-\frac{1}{s^2+1} + 1 = sL\{\cos t\}$$

$$L\{\cos t\} = \frac{-\frac{1}{s^2+1}+1}{s} = \frac{\frac{-1+s^2+1}{s^2+1}}{s}$$

$$= \frac{-1+s^2+1}{s^2+1} \cdot \frac{1}{s} = \frac{s^2}{s(s^2+1)} = \frac{s}{s^2+1}$$

2- **let $f(t), t \geq 0$ be differentiable function then**

$$\boldsymbol{L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)}$$

Proof :

$$\because L\{f'(t)\} = sL\{f(t)\} - f(0)$$

$$\therefore L\{f''(t)\} = sL\{f'(t)\} - f'(0)$$

$$= s[sL\{f(t)\} - f(0) - f'(0)] = s^2 L\{f(t)\} - sf(0) - f'(0)$$

Example : let $\boldsymbol{f(t) = t \sin t}$, find $F(s)$ by using the second derivation

Solution :

$$L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)$$

$$f(t) = t \sin t \;\rightarrow f(0) = 0 \sin 0 = 0$$

$f'(t) = t\cos t + \sin t \to f'(0) = 0\,cos\,0 + sin\,0 = 0$

$f''(t) = -t\sin t + \cos t + cost \ = 2\cos t \ \ - t\sin t$

$\therefore L\{f''(t)\} = L\ \{2\cos t - t\ sint\ \}$

$L\{f''(t)\} = 2\ L\{\cos t\} - \ L\ \{t\sin t\}$

$L\{f''(t)\} = 2\left(\frac{s}{s^2+1}\right) - L\{t\ sint\ \}$ ………………….… (1)

$L\{f''(t)\} = s^2\ L\{t\sin t\} - s(0) - 0 = s^2\ L\{t\sin t\}$ …………….…..…. (2)

Now by (1) and (2) we get :

$$s^2\ L\{t\sin t\} = \frac{2s}{s^2+1} - \ L\{t\sin t\}$$

$$(s^2+1)L\{t\sin t\} = \frac{2s}{s^2+1}$$

$$L\{t\sin t\} = \frac{2s}{(s^2+1)(s^2+1)} = \frac{2s}{(s^2+1)^2}$$

***<u>Note</u>* :**

$\boldsymbol{L\{f'''(t)\} = s^3L\ \{f(t)\} - s^2f(0) - sf'(0) - f''(0)}$

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$$\boldsymbol{L\{f^{n}(t)\} = s^{n}L\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{n-1}(0)}$$

Example : find $\boldsymbol{L\,\{2\ sin\ t\}}$ by using the third derivation

Solution :

$$L\{f'''(t)\} = s^{3}\,L\{f(t)\} - s^{2}f(0) - sf'(0) - f''(0)$$

$$f(t) = 2\sin t \ \rightarrow f(0) = 2\sin(0) = 0$$

$$f'(t) = 2\cos t \ \rightarrow f'(0) = 2\cos(0) = 2(1) = 2$$

$$f''(t) = -2\sin t \ \rightarrow f''(0) = -2\sin(0) = 0$$

$$f'''(t) = -2\cos t$$

$$\therefore L\{f'''(t)\} = L\,\{-2\cos t\} = -2L\,\{\cos t\} = -2\,\frac{s}{s^2+1^2}$$

$$\therefore L\{f'''(t)\} = \frac{-2s}{s^2+1} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1)$$

$$L\{f'''(t)\} = s^{3}L\{f(t)\} - s^{2}f(0) - sf'(0) - f''(0)$$

$$= s^{3}L\{2\sin t\} - s^{2}f(0) - 2s - 0$$

$$\therefore L\{f'''(t)\} = s^{3}L\{2\sin t\} - 2s \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

Now by (1) and (2) we get :

$$\frac{-2s}{s^2+1^2} = s^{3}L\{2\sin t\} - 2s$$

$$\frac{-2s}{s^2+1^2} + 2s = s^{3}L\{2\sin t\}$$

$$\frac{-2s+2s(s^2+1)}{s^2+1^2}=s^3L\{2\sin t\}$$

$$\frac{-2s+2s^3+2s}{s^2+1^2}=s^3L\{2\sin t\}$$

$$\frac{2s^3}{s^2+1^2}=s^3L\{2\sin t\}$$

$$L\{2\sin t\}=\frac{2s^3}{s^2+1}\cdot\frac{1}{s^3}$$

$$L\{2\sin t\}=\frac{2}{(s^2+1)}$$

H.W

1- using the first derivation to find $L\{2t\sin at\}$.

2- **using the second derivation to find $L\{t\cos 3t\}$.**

3- using the third derivation to find $L\{3\cos 5t\}$.

4- let $L\{f(t)\}=F(s)$ then $L\{e^{at}f(t)\}=F(s-a)$

Proof :

$$L\{f(t)\}=\int_0^\infty f(t)\,e^{-st}\,dt$$

$$F(s)=\int_0^\infty f(t)\,e^{-st}\,dt$$

$$F(s-a)=\int_0^\infty f(t)\,e^{-(s-a)t}\,dt=\int_0^\infty f(t)\,e^{-st+at}\,dt$$

$$= \int_{0}^{\infty} f(t)\, e^{-st}\, e^{at}\, dt = \int_{0}^{\infty} e^{-st} \left(e^{at} f(t)\right) dt = L\{e^{at} f(t)\}$$

Example : find $\boldsymbol{L\{e^{3t} \cos 2t\}}$

Solution : $\because L\{e^{at} f(t)\} = F(s-a)$

$$f(t) = \cos 2t \rightarrow L\{f(t)\} = L\{\cos 2t\} = \frac{s}{s^2+4} = F(s)$$

$$L\{e^{at} f(t)\} = L\{e^{3t} \cos 2t\} = F(s-3) = \frac{s-3}{(s-3)^2+4}$$

H.W Find the following:

1- $L\{e^{-at} \sin bt\}$

2- $L\{4e^{t} \cos 2t\}$

5- $\boldsymbol{L\left\{\int_0^t f(t)\, dt\right\} = \frac{L\{f(t)\}}{s} = \frac{F(s)}{s}}$

Example : find $\boldsymbol{L\left\{\int_0^t cost\, dt\right\}}$

Solution : $L\left\{\int_0^t f(t)\, dt\right\} = \frac{L\{f(t)\}}{s} = \frac{F(s)}{s}$

Now $f(t) = cost \rightarrow F(s) = L\{f(t)\} = L\{cost\} = \frac{s}{s^2+1}$

$$L\left\{\int_0^t cost\, dt\right\} = \frac{L\{cost\}}{s} = \frac{\frac{s}{s^2+1}}{s} = \frac{s}{s(s^2+1)} = \frac{1}{s^2+1}$$

H.W **1- Evaluate the following** $L\left\{\int_0^t 2sin3t\ dt\right\}$, $L\left\{\int_0^t e^{3t}cost\ dt\right\}$

2- compute $L\{e^{-2t}\sin 5t + 3tcos\ 2t\}$

3- Evaluate $L\{2e^t - e^{4t}\cos t\}$

4- Find $F(s)$ **where** $f(t) = t(2t+3)(t-8)$

5- Let $f(t) = 3t\ cosh\ 2t$ **, find** $L\{f''(t)\}$

6- Evaluate $L\{t^2e^{3t}\}$ **.**

7- Compute $L\{\cos^2 a\,t\}, a$ **any constant.**

8- Evaluate $L\{\sin^2 a\,t\}, a$ **any constant.**

9- Find $L\{\cos^2 t\}$ **.**

10- Compute $L\{\sin^2 3\,t\}$ **.**

11- Evaluate $L\{t(t+1)(t+2)\}$