Mathematical Transform

Chapter One : (Laplace Transforms)

1- Definition of Laplace Transform.
2- Laplace Transform of some special functions.
3- Properties of Laplace Transform.
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5- Invers Laplace Transform.
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3- Half Range Fourier series.
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Chapter One : (Laplace Transforms)

1- Definition of Laplace Transform :

Let $f(t),\ t \geq 0$ be any function . The Laplace Transform of $f(t)$ is denoted by $F(s)$, $s > 0$ and define as following :

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \lim_{A\to\infty} \int_0^A e^{-st} f(t)dt$$

Example : by using the definition of Laplace Transform find $L\left\{\frac{1}{3}\right\}$.

Solution: :

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \lim_{A\to\infty} \int_0^A e^{-st} f(t)dt$$

$$F(s) = L\left\{\frac{1}{3}\right\} = \lim_{A\to\infty} \int_0^A e^{-st}\ \frac{1}{3}dt = \frac{1}{3}\lim_{A\to\infty} \int_0^A e^{-st}\ dt$$

$$= \frac{\frac{1}{3}}{-s} \lim_{A\to\infty} \int_0^A -s\ e^{-st}\ dt$$

$$= \frac{1}{-3s} \lim_{A\to\infty} [e^{-st}]_0^A = \frac{1}{-3s}\left(\lim_{A\to\infty} e^{-sA} - e^{-0}\right)$$

$$= \frac{1}{-3s}\ (e^{-\infty} - 1) = \frac{1}{3s}\ , s > 0$$

H.W

By using the definition of Laplace Transform find $L\{\ a\ \}$, a any scalar.

2- Laplace Transform of some special functions :

1- If $\boldsymbol{f(t)=1}$, then $\boldsymbol{F(s)=L\{f(t)\}=L\{1\}=\frac{1}{s}}$ $\quad$ $\boldsymbol{,s>0}$

Proof :

$$F(s)=L\{f(t)\}=\int_0^\infty e^{-st}\,f(t)dt=\lim_{A\to\infty}\int_0^A e^{-st}\,f(t)dt$$

$$F(s)=L\{1\}=\lim_{A\to\infty}\int_0^A e^{-st}\,1dt=1\lim_{A\to\infty}\int_0^A e^{-st}\ dt$$

$$=\frac{1}{-s}\lim_{A\to\infty}\int_0^A -s\,e^{-st}\ dt\ =\frac{1}{-s}\lim_{A\to\infty}[e^{-st}]_0^A=\frac{1}{-s}\left(\lim_{A\to\infty}e^{-sA}-e^{-0}\right)$$

$$=\frac{1}{-s}\left(e^{-\infty}-1\right)=\frac{1}{s}\ ,s>0$$

2- If $\boldsymbol{f(t)=t\ ,t\geq 0}$, then $\boldsymbol{F(s)=L\{f(t)\}=L\{t\}=\frac{1}{s^2}}$, $\boldsymbol{s>0}$

Proof :

$$L\{f(t)\}=\int_0^\infty e^{-st}\ f(t)\,dt=\lim_{A\to\infty}\int_0^A e^{-st}\,f(t)\,dt$$

$$L\{t\}=\lim_{A\to\infty}\int_0^A e^{-st}\,t\,dt$$

تكامل بطريقة udv

$$u=t\to du=dt$$

$$dv=e^{-st}\,dt\to v=\frac{1}{-s}e^{-st}$$

$$\boxed{\lim_{A\to\infty}\int_0^A u\,dv = \lim_{A\to\infty} uv\Big]_0^A - \lim_{A\to\infty}\int_0^A v\,du}$$

$$\therefore \lim_{A\to\infty}\int_0^A e^{-st}\,t\,dt = \lim_{A\to\infty}\left[\frac{te^{-st}}{-s}\right]_0^A - \lim_{A\to\infty}\int_0^A \frac{e^{-st}}{-s}\,dt$$

$$= \frac{1}{-s}\left(\lim_{A\to\infty} Ae^{-sA} - 0\right) - \left(\frac{1}{-s}\right)\lim_{A\to\infty}\int_0^A e^{-st}\,dt$$

$$= \frac{1}{-s}(0-0) + \frac{1}{s}\left(\frac{1}{-s}\right)\lim_{A\to\infty}\int_0^A -se^{-st}dt = \frac{-1}{s^2}\lim_{A\to\infty}[e^{-st}]_0^A$$

$$= \frac{-1}{s^2}\left(\lim_{A\to\infty} e^{-sA} - e^0\right) = \frac{1}{-s^2}[0-1] = \frac{1}{s^2}$$

3- If $\boldsymbol{f(t) = t^n}$, $\boldsymbol{t \geq 0, n \in N}$, then $\boldsymbol{F(s) = L\{f(t)\} = L\{t^n\} = \frac{n!}{s^{n+1}}}$

Proof :

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \lim_{A\to\infty}\int_0^A e^{-st} f(t)\,dt$$

$$L\{t^n\} = \lim_{A\to\infty}\int_0^A e^{-st}\,t^n\,dt = \lim_{A\to\infty}\int_0^A e^{-st}\,t^n\,dt$$

تكامل بطريقة udv

$u = t^n \to du = nt^{n-1}dt$

$dv = e^{-st}\,dt \to v = \frac{e^{-st}}{-s}$

$$\because \lim_{A\to\infty}\int_0^A u\,dv = \lim_{A\to\infty} uv\Big]_0^A - \lim_{A\to\infty}\int_0^A v\,du$$

$$\therefore L\{t^n\} = \lim_{A\to\infty}\left[\frac{-t^n e^{-st}}{s}\right]_0^A - \lim_{A\to\infty}\int_0^A \frac{-n}{s} e^{-st}\, t^{n-1}\,dt$$

$$= \left(\lim_{A\to\infty}\frac{-A^n}{se^{sA}} - 0\right) - \left(\frac{-n}{s}\right)\lim_{A\to\infty}\int_0^A e^{-st}\, t^{n-1}\,dt$$

حسب قاعدة لوبيتال

$$= \frac{n}{s}\int_0^\infty e^{-st}\, t^{n-1}\,dt = \frac{n}{s} L\{t^{n-1}\}$$

$$= \frac{n}{s}\left[\frac{n-1}{s} L\{t^{n-2}\}\right] = \frac{n}{s} \cdot \frac{n-1}{s} L\{t^{n-2}\}$$

$$= \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdot L\{t^{n-3}\}$$

$$= \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdot \ldots \cdot \frac{2}{s} \cdot \frac{1}{s}\, L\{t^0\}$$

$$= \frac{(n)(n-1)\ldots(2)(1)}{s^n} \cdot L\{1\}$$

$$= \frac{n!}{s^n} \cdot \frac{1}{s} == \frac{n!}{s^{n+1}}$$

Example :1- find Laplace Transform of $f(t) = t^3$.

2- find Laplace Transform of $f(t) = t^{7-a}$, a any scalar.

Solution:

1- $F(s) = L\{f(t)\} = L\{t^3\} = \frac{3!}{s^4} = \frac{3*2*1}{s^4} = \frac{6}{s^4}$

2- $F(s) = L\{f(t)\} = L\{t^{7-a}\} = \frac{(7-a)!}{s^{(7-a)+1}}$

$= \frac{(7-a)(7-a-1)(7-a-2)(7-a-3)\dots(3)(2)(1)}{s^{8-a}} =$

$\frac{(7-a)(6-a)(5-a)(4-a)\dots(3)(2)(1)}{s^{8-a}}$

4- If $f(t) = e^{at}$, a any scalar , then $F(s) = L\{e^{at}\} = \frac{1}{s-a}$, $s > a$.

Proof : $L\{f(t)\} = \int_0^\infty e^{-st}\ f(t)\, dt = lim_{A\to\infty} \int_0^A e^{-st}\, f(t)\, dt$

$$L\{e^{at}\} = \lim_{A\to\infty} \int_0^A e^{-st} e^{at}\ dt = \lim_{A\to\infty} \int_0^A e^{(-s+a)t}\ dt = \lim_{A\to\infty} \int_0^A e^{-(s-a)t}\ dt$$

$$= \frac{-1}{(s-a)} \lim_{A\to\infty} \int_0^\infty (-(s-a))\, e^{-(s-a)t}\ dt$$

$$= \frac{-1}{(s-a)} \lim_{A\to\infty} \left[e^{-(s-a)t}\right]_0^A$$

$$= \frac{-1}{(s-a)} \left[\lim_{A\to\infty} e^{-(s-a)A} - e^0\right]$$

$$= \frac{-1}{(s-a)} [e^{-\infty} - 1] = \frac{1}{s-a}$$

Example :1- find Laplace Transform of $f(t) = e^{3t}$.

2- find Laplace Transform of $f(t) = e^{-2t}$.

Solution :

1- $F(s) = L\{f(t)\} = \frac{1}{s-3}$

2- $F(s) = L\{f(t)\} = \frac{1}{s+2}$

5- If $\boldsymbol{f(t) = \sin at, , t \geq 0, a}$ any scalar

$$\boldsymbol{F(s) = L\{f(t)\} = L\{\sin at\} = \frac{a}{s^2 + a^2}}$$

Proof : $L\{f(t)\} = \int_0^\infty e^{-st}\ f(t)\,dt = lim_{A\to\infty} \int_0^A e^{-st}\, f(t)\,dt$

$$L\{\sin at\} = \int_0^\infty e^{-st} \sin at\, dt$$

D	I
e^{-st}	$\sin at$
$-s\ e^{-st}$	$\frac{-1}{a}\cos at$
$s^2\ e^{-st}$	$\frac{-1}{a^2}\sin at$

$$\int_0^\infty \sin at\, e^{-st}\, dt = \left[\frac{-e^{-st}}{a}\cos at - \frac{se^{-st}}{a^2}\ \sin at\right]_0^\infty - \frac{s^2}{a^2}\int_0^\infty \sin at\, e^{-st}\ dt$$

$$\left(1 + \frac{s^2}{a^2}\right)\int_0^\infty \sin at\, e^{-st} = \lim_{A\to\infty}\left[\frac{-e^{-st}}{\mathrm{a}}\cos at - \frac{se^{-st}}{a^2}\ \sin at\right]_0^A$$

$$\left(\frac{a^2+s^2}{a^2}\right)\int_0^\infty \sin at\, e^{-st} = \lim_{A\to\infty}\left[\frac{-e^{-st}}{\mathrm{a}}\cos at\right]_0^A - \lim_{A\to\infty}\left[\frac{se^{-st}}{a^2}\ \sin at\right]_0^A$$

$$\left(\frac{a^2+s^2}{a^2}\right)\int_0^\infty sin\, at\, e^{-st} = \frac{-1}{a}\left[\lim_{A\to\infty} e^{-sA} \cos aA - e^{-s(0)} \cos(0)\right] - \frac{s}{a^2}\left[\lim_{A\to\infty} e^{-sA} \sin aA - 0\right]$$

$$\left(\frac{a^2+s^2}{a^2}\right)\int_0^\infty sin\, at\, e^{-st} = \frac{-1}{a}\,[0-1] - 0 = \frac{1}{a}$$

$$\int_0^\infty sin\, at\, e^{-st} dt = \frac{\frac{1}{a}}{\left(\frac{a^2+s^2}{a^2}\right)} = \frac{1}{a} * \frac{a^2}{a^2+s^2} = \frac{a}{a^2+s^2}$$

Example :1- find Laplace Transform of $f(t) = sin\, 3t$.

2- find Laplace Transform of $f(t) = sin(-5)t$

Solution :

1- $F(s) = L\{sin\, 3t\} = \frac{3}{s^2+3^2} = \frac{3}{s^2+9}$

2- $F(s) = L\{sin(-5)t\} = \frac{-5}{s^2+(-5)^2} = \frac{-5}{s^2+25}$

6 - If $f(t) = \cos at$, $t \geq 0$, a any scalar, then

$$\boldsymbol{F(s) = L\{f(t)\} = L\{cas\ at\} = \frac{s}{s^2+a^2}}$$

Proof : ***H.W***

Example :1- find Laplace Transform of $f(t) = cos\, 4t$.

2- find Laplace Transform of $f(t) = cos(-7)t$.

Solution : 1- $F(s) = L\{cos\, 4t\} = \frac{s}{s^2+16}$

2- $F(s) = L\{cos(-7)t\} = \frac{s}{s^2+(-7)^2} = \frac{s}{s^2+49}$

7- If $\boldsymbol{f(t) = \sinh at}$, $\boldsymbol{t \geq 0, a}$ any scalar , then

$$\boldsymbol{F(s) = L\{f(t)\} = L\{\sinh at\} = \frac{a}{s^2 - a^2}}$$

Proof : ***H.W***

Example : 1- find Laplace Transform of $f(t) = \sinh 2t$.

2- find Laplace Transform of $f(t) = \sinh(-3b)t, b$ any scalar .

Solution:

1- $F(s) = L\{\sinh 2t\} = \frac{2}{s^2-4}$

2- $F(s) = L\{\sinh(-3b)t\} = \frac{-3b}{s^2-(-3b)^2} = \frac{-3b}{s^2-9b^2}$

8- If $\boldsymbol{f(t) = \cosh at}$, $\boldsymbol{t \geq 0, a}$ any scalar then

$$\boldsymbol{F(s) = L\{f(t)\} = L\{\cosh at\} = \frac{s}{s^2 - a^2}}$$

Proof : $L\{f(t)\} = \int_0^\infty e^{-st}\ f(t)\ dt = lim_{A\to\infty} \int_0^A e^{-st}\ f(t)\ dt$

$L\{\cosh at\} = \int_0^\infty e^{-st} \cosh at\ dt$

D	I
e^{-st}	$\cosh at$
$-s\ e^{-st}$	$\frac{1}{a}\sinh at$
$s^2\ e^{-st}$	$\frac{1}{a^2}\cosh at$

$$\int_0^\infty \cosh at\, e^{-st}\, dt = \left[\frac{1}{a}e^{-st}\sinh at + \frac{s}{a^2}\, e^{-st}\cosh at\right]_0^\infty + \frac{s^2}{a^2}\int_0^\infty \cosh at\, e^{-st}\, dt$$

$$\left(1 - \frac{s^2}{a^2}\right)\int_0^\infty \cosh at\, e^{-st}\, dt = \left[\frac{1}{a}e^{-st}\sinh at + \frac{s}{a^2}\, e^{-st}\cosh at\right]_0^\infty$$

$$\left(\frac{a^2 - s^2}{a^2}\right)\int_0^\infty \cosh at\, e^{-st} dt = \lim_{A\to\infty}\left[\frac{1}{a}e^{-st}sinhat + \frac{s}{a^2}\, e^{-st}\cosh at\right]_0^A$$

$$\left(\frac{a^2 - s^2}{a^2}\right)\int_0^\infty \cosh at\, e^{-st} dt = \lim_{A\to\infty}\left[\frac{1}{a}e^{-st}sinhat\right]_0^A + \lim_{A\to\infty}\left[\frac{s}{a^2}\, e^{-st}\cosh at\right]_0^A$$

$$= \frac{1}{a}\left[\lim_{A\to\infty} e^{-sA}sinha(A) - e^{-(0)}\sinh(0)\right]$$

$$+ \frac{s}{a^2}\left[\lim_{A\to\infty} e^{-sA}cosha(A) - e^{-(0)}\cosh(0)\right]$$

$$\left(\frac{a^2 - s^2}{a^2}\right)\int_0^\infty \cosh at\, e^{-st} dt = (0 - 0) + \frac{s}{a^2}(0 - 1)$$

$$\left(\frac{a^2 - s^2}{a^2}\right)\int_0^\infty \cosh at\, e^{-st} dt = -\frac{s}{a^2}$$

$$\int_0^\infty \cosh at\, e^{-st} dt = \frac{-\frac{s}{a^2}}{\frac{a^2 - s^2}{a^2}} = -\frac{s}{a^2} * \frac{a^2}{a^2 - s^2} = -\frac{s}{-(s^2 - a^2)}$$

$$= \frac{s}{s^2 - a^2}$$

Example :1- find Laplace Transform of $f(t) = cosh\,3t$.

2- find Laplace Transform of $f(t) = cosh(-6)t$.

Solution:

1- $F(s) = L\{cosh\,3t\} = \frac{s}{s^2-9}$

2- $F(s) = L\{cosh\,3t\} = \frac{s}{s^2-36}$

Now we write Laplace Transform of some special functions in the following table :

	$f(t)$	$F(s)$
1	1	$\frac{1}{s}$
2	t	$\frac{1}{s^2}$
3	t^n	$\frac{n!}{s^{n+1}}$
4	e^{at}	$\frac{1}{s-a}$
5	$sinat$	$\frac{a}{s^2+a^2}$
6	$cosat$	$\frac{s}{s^2+a^2}$
7	$\sinh at$	$\frac{a}{s^2-a^2}$
8	$cosh\,at$	$\frac{s}{s^2-a^2}$

H.W

1- Find Laplace Transform of the following :

a- $f(t) = t^{3b}$, b any scalar .

b- $f(t) = sin5t$.

c- $f(t) = \cos 9t$.

d- $f(t) = sinh\, t$.

f - $f(t) = e^{8t}$

2- By using the definition of Laplace Transform find $F(s) = L\{f(t)\}$ where :

a- $f(t) = t^{4b}$, b any scalar .

b- $f(t) = sin7t$.

c- $f(t) = cosh\, t$.

d- $f(t) = e^{5t}$

H.W

6 - let $\boldsymbol{f(t)} = \cos \boldsymbol{at}$, $\boldsymbol{t} \geq 0$, $\boldsymbol{a}$ any scalar, then

$$\boldsymbol{F(s) = L\{f(t)\} = L\{cas\ at\} = \frac{s}{s^2 + a^2}}$$

Proof :

$$L\{f(t)\} = \int_0^\infty e^{-st}\ f(t)\ dt = \lim_{A\to\infty} \int_0^A e^{-st}\ f(t)\ dt$$

$$L\{cas\ at\} = \int_0^\infty e^{-st} \cos at\ dt$$

D	I
e^{-st}	$\cos at$
$-s\ e^{-st}$	$\frac{1}{a} \sin at$
$s^2\ e^{-st}$	$\frac{-1}{a^2} \cos at$

$$\int_0^\infty \cos at\ e^{-st}\ dt = \left[\frac{1}{a} e^{-st} \sin at - \frac{s}{a^2}\ e^{-st} \cos at\right]_0^\infty - \frac{s^2}{a^2} \int_0^\infty e^{-st} \cos at$$

$$\left(1 + \frac{s^2}{a^2}\right) \int_0^\infty e^{-st} \cos at = \frac{1}{a}\left[\lim_{A\to\infty} e^{-sA} \sin aA - e^{-s(0)} \sin(0)\right]$$

$$-\frac{s}{a^2}\left[\lim_{A\to\infty} e^{-sA} \cos aA - e^{-s(0)} \cos(0)\right]$$

$$\left(1+\frac{s^2}{a^2}\right)\int_0^\infty e^{-st}\cos at = \frac{1}{a}[0-0] - \frac{s}{a^2}[0-1]$$

$$1+\frac{s^2}{a^2}\int_0^\infty e^{-st}\cos at = \frac{s}{a^2}$$

$$\int_0^\infty \sin at\ e^{-st}at = \frac{\frac{s}{a^2}}{1+\frac{s^2}{a^2}} = \frac{\frac{s}{a^2}}{\frac{a^2+s^2}{a^2}} = \frac{s}{a^2}\cdot\frac{a^2}{a^2+s^2} = \frac{s}{s^2+a^2}$$

7- **let $\boldsymbol{f(t)=\sinh at}$, $\boldsymbol{t\geq 0, a}$ any scalar , then**

$$\boldsymbol{F(s) = L\{f(t)\} = L\{\sinh at\} = \frac{a}{s^2-a^2}}$$

Proof : $L\{f(t)\} = \int_0^\infty e^{-st}\ f(t)\,dt = lim_{A\to\infty}\int_0^A e^{-st}\,f(t)\,dt$

$L\{\sinh at\} = \int_0^\infty e^{-st}\sinh at\,dt$

D	I
e^{-st}	$\sinh at$
$-s\ e^{-st}$	$\frac{1}{a}\cosh at$
$s^2\ e^{-st}$	$\frac{-1}{a^2}\sinh at$

$$\int_0^\infty \sinh at\ e^{-st}\ dt = \left[\frac{1}{a}e^{-st}\cosh at + \frac{s}{a^2}\ e^{-st}\sin at\right]_0^\infty + \frac{s^2}{a^2}\int_0^\infty \sinh at\ e^{-st}\ dt$$

$$\left(1-\frac{s^2}{a^2}\right)\int_0^\infty \sinh at\ e^{-st}dt = \lim_{A\to 0}\left[\frac{1}{\mathrm{a}}e^{-st}\cosh at + \frac{s}{\mathrm{a}^2}\ e^{-st}\sinh at\right]_0^A$$

$$\left(1-\frac{s^2}{a^2}\right)\int_0^\infty \sinh at\, e^{-st}dt = \frac{1}{\mathrm{a}}\left[\lim_{A\to\infty} e^{-sA}\cosh A - e^{-(0)}\cosh(0)\right]$$

$$+\frac{s}{a^2}\left[\lim_{A\to\infty} e^{-st}\sinh at\, A - e^{-s(0)}\sin(0)\right]$$

$$\left(1-\frac{s^2}{a^2}\right)\int_0^\infty \sinh at\, e^{-st}dt = \frac{1}{\mathrm{a}}(0-1)+\frac{s}{a^2}(0-0)$$

$$\left(1-\frac{s^2}{a^2}\right)\int_0^\infty \sinh at\, e^{-st}dt = \frac{-1}{a}$$

$$\int_0^\infty \sinh at\, e^{-st}dt = \frac{-\frac{1}{a}}{1-\frac{s^2}{a^2}} = -\frac{1}{a}*\frac{a^2}{a^2-s^2} = \frac{-a}{a^2-s^2} = \frac{-a}{-(-a^2+s^2)} = \frac{a}{s^2-a^2}$$