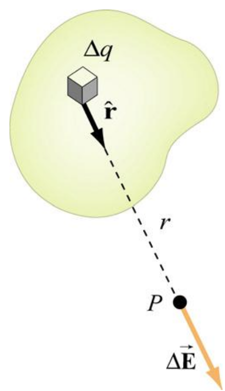
Continuous Charge Distribution

If the charge distribution is continuous, the electric potential at a point *P due to dq is*

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r}$$

Summing over contributions from all differential elements, we have

$$V = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{r}$$



Deriving Electric Field from the Electric Potential

If we consider two points which are separated by a small distance $d\mathbf{s}$, the following differential form is obtained:

$$dV = -\vec{\mathbf{E}}\cdot d\vec{\mathbf{s}}$$

In Cartesian coordinate

$$\vec{\mathbf{E}} = E_x\hat{\mathbf{i}} + E_y\hat{\mathbf{j}} + E_z\hat{\mathbf{k}}$$

$$d\vec{\mathbf{s}} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}$$

$$dV = \left(E_x\hat{\mathbf{i}} + E_y\hat{\mathbf{j}} + E_z\hat{\mathbf{k}}\right)\cdot\left(dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}\right) = E_x dx + E_y dy + E_z dz$$

Which implies

$$E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z}$$

A differential quantity called the “del (gradient) operator”

$$\nabla \equiv \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}$$

the electric field can be written as

$$\vec{\mathbf{E}} = E_x\hat{\mathbf{i}} + E_y\hat{\mathbf{j}} + E_z\hat{\mathbf{k}} = -\left(\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}\right) = -\left(\frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}\right)V = -\nabla V$$

$$\vec{\mathbf{E}} = -\nabla V$$

Mathematically meaning: The $\vec{E}$ as the negative of the *gradient* of the electric potential *V*.

Physically meaning: The negative sign implies that if *V* increases as a positive charge moves along some direction, say *x*, with $\frac{\partial V}{\partial x} > 0$, then there is a non-vanishing component of $\vec{E}$ in the opposite direction ($-E_x \neq 0$)

<u>Note</u>: ∇ operates on a scalar quantity (electric potential) and results in a vector quantity (electric field).

If the charge distribution possesses spherical symmetry, then the resulting electric field is a function of the radial distance *r*,

i.e.

$$\vec{\mathbf{E}} = E_r\hat{\mathbf{r}}$$

$$dV = -E_r\, dr$$

$$\vec{\mathbf{E}} = E_r \hat{\mathbf{r}} = -\left(\frac{dV}{dr}\right)\hat{\mathbf{r}}$$

For example, the electric potential due to a point charge q is

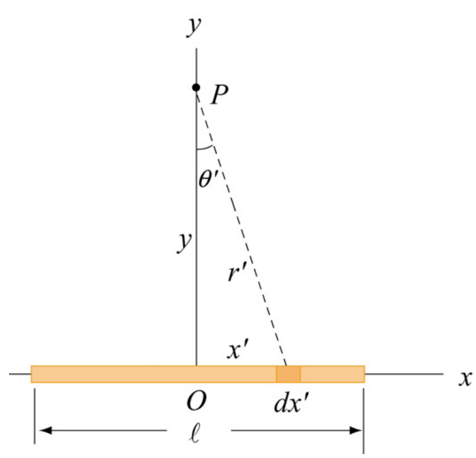
$$V(r) = q / 4\pi\varepsilon_0 r$$

The electric field is simply

$$\vec{\mathbf{E}} = (q/4\pi\varepsilon_0 r^2)\hat{\mathbf{r}}$$

Example

Consider a non-conducting rod of length l having a uniform charge density λ. Find the electric potential at P, a perpendicular distance y above the midpoint of the rod.



Solution:

a differential element of length dx' which carries a charge $dq = \lambda\, dx'$.

The distance from dx' to P is

$$r = (\acute{x}^2 + y^2)^{1/2}$$

the potential is given by

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dx'}{(x'^2+y^2)^{1/2}}$$

the total potential due to the entire rod is

$$V = \frac{\lambda}{4\pi\varepsilon_0}\int_{-\ell/2}^{\ell/2}\frac{dx'}{\sqrt{x'^2+y^2}} = \frac{\lambda}{4\pi\varepsilon_0}\ln\left[x'+\sqrt{x'^2+y^2}\right]\Bigg|_{-\ell/2}^{\ell/2}$$
$$= \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{(\ell/2)+\sqrt{(\ell/2)^2+y^2}}{-(\ell/2)+\sqrt{(\ell/2)^2+y^2}}\right]$$

where we have used the integration formula

$$\int\frac{dx'}{\sqrt{x'^2+y^2}} = \ln\left(x'+\sqrt{x'^2+y^2}\right)$$

In the limit $l \gg y$ the potential becomes,

$$V = \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{(\ell/2)+\ell/2\sqrt{1+(2y/\ell)^2}}{-(\ell/2)+\ell/2\sqrt{1+(2y/\ell)^2}}\right] = \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{1+\sqrt{1+(2y/\ell)^2}}{-1+\sqrt{1+(2y/\ell)^2}}\right]$$
$$\approx \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{2}{2y^2/\ell^2}\right) = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{\ell^2}{y^2}\right)$$
$$= \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{\ell}{y}\right)$$

The corresponding electric field can be obtained as

$$E_y = -\frac{\partial V}{\partial y} = \frac{\lambda}{2\pi\varepsilon_0 y}\frac{\ell/2}{\sqrt{(\ell/2)^2+y^2}}$$

H.W.

1. Suppose in some region of space the electric potential is given by

$$V(x,y,z) = V_0 - E_0 z + \frac{E_0 a^3 z}{(x^2 + y^2 + z^2)^{3/2}}$$

Where a is a constant with dimensions of length. Find the *x*, *y*, and the *z*-components of the associated electric field.

2. Suppose that the electric potential in some region of space is given by

$$V(x,y,z) = V_0 \exp(-k\,|\,z\,|) \cos kx$$

Find the electric field everywhere.