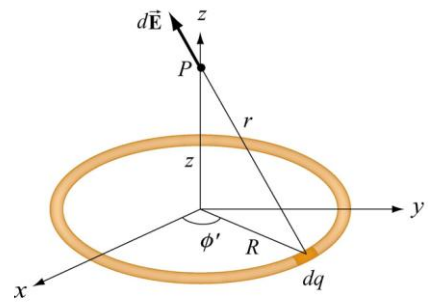
***Example*:** A non-conducting ring of radius *R* with a uniform charge density λ and a total charge *Q* is lying in the *xy*- plane, as shown in Figure. Compute the electric field at a point *P*, located at a distance *z* from the center of the ring along its axis of symmetry.

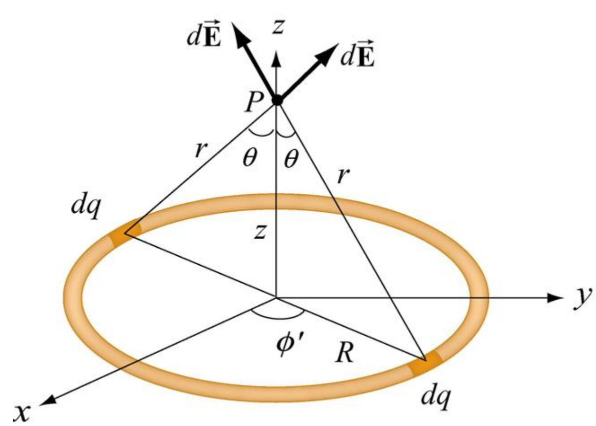


Solution: The contribution to the electric field at ***P*** from a small length element $d\acute{l}$ on the ring is

$$d\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\hat{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda R\, d\phi'}{r^2}\hat{\mathbf{r}}$$

Where $dq = \lambda\, d\acute{l} = \lambda\, R\, d\acute{\phi}$

The *y*-component of the electric field (E_y) vanishes (symmetry) as shown in figure



The electric field at P must point in the +Z direction.

$$dE_z = dE\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{\lambda R\,d\phi'}{R^2+z^2}\frac{z}{\sqrt{R^2+z^2}} = \frac{\lambda}{4\pi\varepsilon_0}\frac{Rz\,d\phi'}{(R^2+z^2)^{3/2}}$$

Where $cos\theta = \frac{z}{r} = \frac{z}{\sqrt{R^2+z^2}}$

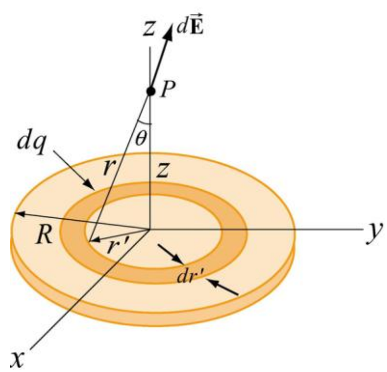
The total electric field due to the ring is

$$E_z = \frac{\lambda}{4\pi\varepsilon_0}\frac{Rz}{(R^2+z^2)^{3/2}}\oint d\phi' = \frac{\lambda}{4\pi\varepsilon_0}\frac{2\pi Rz}{(R^2+z^2)^{3/2}} = \frac{1}{4\pi\varepsilon_0}\frac{Qz}{(R^2+z^2)^{3/2}}$$

where the total charge is $Q = \lambda(2\pi R)$.

Note: ***The electric field at the center of the ring (z = 0) vanishes. This is to be expected from symmetry arguments.***

Example: A uniformly charged disk of radius R with a total charge Q lies in the xy-plane. Find the electric field at a point P, along the z-axis that passes through the center of the disk perpendicular to its plane. Discuss the limit where $R \gg z$.



Solution:

Consider a ring of radius r' and thickness dr'.

The y-component of the electric field vanishes (symmetry).

The electric field at P points in the +z-direction

$$d\bar{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\hat{\mathbf{r}}$$

$$dE_z = dE\cos\theta$$ **Where** $\cos\theta = \frac{z}{r} = \frac{z}{\sqrt{\acute{r}^2+z^2}}$

$$dE_z = \frac{1}{4\pi\varepsilon_0}\frac{z\,dq}{(r'^2+z^2)^{3/2}} = \frac{1}{4\pi\varepsilon_0}\frac{z(2\pi\sigma r'\,dr')}{(r'^2+z^2)^{3/2}}$$

Where $dq = \sigma(2\pi r'\,dr')$

Integrating from $\acute{r}=0$ to $\acute{r}=R$, the total electric field at P becomes

$$E_z = \int dE_z = \frac{\sigma z}{2\varepsilon_0}\int_0^R \frac{r'\,dr'}{(r'^2+z^2)^{3/2}}$$

$$= -\frac{\sigma z}{2\varepsilon_0}\left[\frac{1}{\sqrt{R^2+z^2}} - \frac{1}{\sqrt{z^2}}\right] = \frac{\sigma}{2\varepsilon_0}\left[\frac{z}{|z|} - \frac{z}{\sqrt{R^2+z^2}}\right]$$

The above equation may be rewritten as

$$E_z = \begin{cases} \dfrac{\sigma}{2\varepsilon_0}\left[1 - \dfrac{z}{\sqrt{z^2+R^2}}\right], & z > 0 \\ \dfrac{\sigma}{2\varepsilon_0}\left[-1 - \dfrac{z}{\sqrt{z^2+R^2}}\right], & z < 0 \end{cases}$$

1. If $z \gg R$, by using Taylor-series expansion

$$1 - \frac{z}{\sqrt{z^2+R^2}} = 1 - \left(1 + \frac{R^2}{z^2}\right)^{-1/2} = 1 - \left(1 - \frac{1}{2}\frac{R^2}{z^2} + \cdots\right) \approx \frac{1}{2}\frac{R^2}{z^2}$$

The electric field in this limit becomes,

$$E_z = \frac{\sigma}{2\varepsilon_0}\frac{R^2}{2z^2} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma\pi R^2}{z^2} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{z^2}$$

- ***This is indeed the expected "point-charge" result.***

2. If $z \ll R$, The electric field in this limit becomes, in unit-vector notation,

$$\vec{\mathbf{E}} = \begin{cases} \dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}}, & z > 0 \\ -\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}}, & z < 0 \end{cases}$$

- ***Physically this means that the plane is very large, or the field at point P is extremely close to the surface of the plane.***