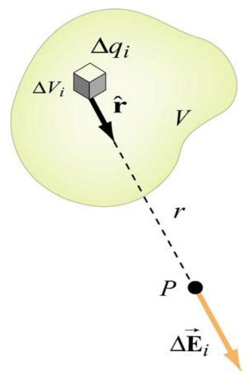
Charge Density

Volume Charge Density

- The volume charge density tells specifically about how much electric charge is present in a given volume.
- We have a very large number of charges distributed in some region in space.
- A small volume element ΔV_i which contains an amount of charge Δq_i.
- r is the distance between ΔV_i and P.



- To find the electric field at some point P.
  - If the distance between charges within the volume element $\Delta V_i \ll r$.
    - $\Delta V_i \rightarrow 0$(infinitesimally small)
    - a volume charge density $\rho(\vec{r})$[C/m^3] is

$$\rho(\vec{\mathbf{r}}) = \lim_{\Delta V_i \to 0} \frac{\Delta q_i}{\Delta V_i} = \frac{dq}{dV}$$

    - The total amount of charge within the entire volume V is

$$Q = \sum_i \Delta q_i = \int_V \rho(\vec{\mathbf{r}})\, dV$$

Surface Charge Density

- the charge can be distributed over a surface S of area A with a surface charge density σ (C/m^2)

$$\sigma(\vec{r}) = \frac{dq}{dA}$$

  - The total charge on the entire surface is:

$$Q = \iint_S \sigma(\vec{\mathbf{r}})\, dA$$

Line Charge Density

If the charge is distributed over a line of length *l*, then the *linear charge density* λ (C/m) is

$$\lambda(\vec{\mathbf{r}}) = \frac{dq}{d\ell}$$

The total charge is now an integral over the entire length:

$$Q = \int_{\text{line}} \lambda(\vec{\mathbf{r}})\, d\ell$$

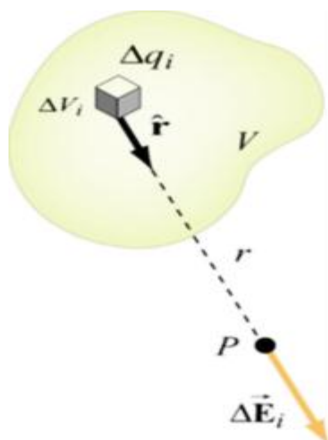
Electric Fields due to Continuous Charge Distributions

The electric field at a point *P* due to each charge element *dq* is given by Coulomb's law:

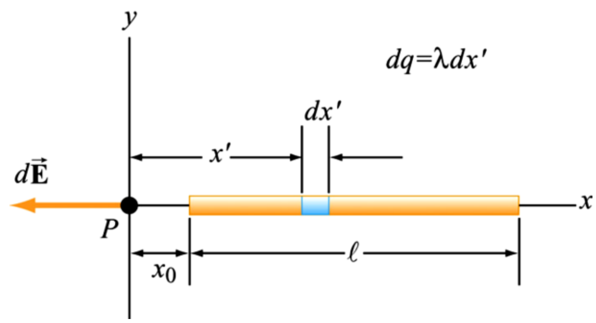
$$d\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\hat{\mathbf{r}}$$

The total electric field $\vec{E}$ is the vector sum (integral) of all these infinitesimal contributions:

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\int_V \frac{dq}{r^2}\hat{\mathbf{r}}$$



Example: A non-conducting rod of length l with a uniform positive charge density λ and a total charge Q is lying along the x-axis, as illustrated in Figure.



- Calculate the electric field at a point P located along the axis of the rod and a distance x_0 from one end.

Solution:

- The linear charge density is uniform and is given by $\lambda = Q/l$.
- The amount of charge contained in a small segment of length $d\grave{x}$ is $dq = \lambda\, d\grave{x}$
- The field at P points in the negative x direction (Since the source carries a positive charge Q).
- The unit vector that points from the source to P is $\hat{r} = -\hat{\imath}$.
  - The contribution to the electric field due to is dq

$$d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\hat{r} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda dx'}{x'^2}(-\hat{i}) = -\frac{1}{4\pi\varepsilon_0}\frac{Q dx'}{\ell x'^2}\hat{i}$$

Integrating over the entire length leads to

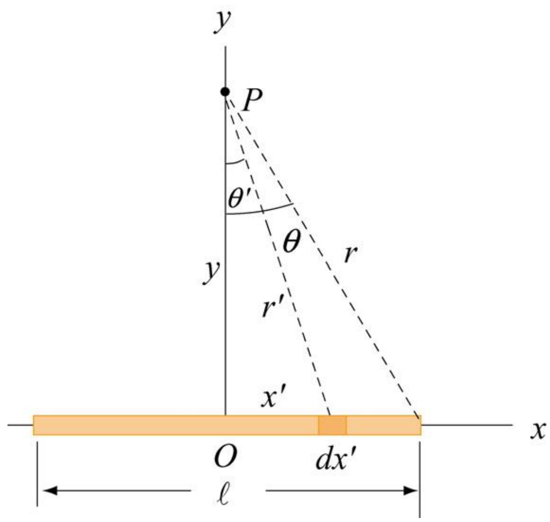
$$\vec{E} = \int d\vec{E} = -\frac{1}{4\pi\varepsilon_0}\frac{Q}{\ell}\int_{x_0}^{x_0+\ell}\frac{dx'}{x'^2}\hat{i} = -\frac{1}{4\pi\varepsilon_0}\frac{Q}{\ell}\left(\frac{1}{x_0} - \frac{1}{x_0+\ell}\right)\hat{i} = -\frac{1}{4\pi\varepsilon_0}\frac{Q}{x_0(\ell + x_0)}\hat{i}$$

- If P is very far away from the rod ($x_0 \gg l$),

$$\vec{\mathbf{E}} \approx -\frac{1}{4\pi\varepsilon_0}\frac{Q}{x_0^2}\hat{\mathbf{i}}$$

***Note:* The result is to be expected since at sufficiently far distance away, the distinction between a continuous charge distribution and a point charge diminishes.**

Example: A non-conducting rod of length l with a uniform charge density λ and a total charge Q is lying along the x-axis, as illustrated in Figure. Compute the electric field at a point P, located at a distance y from the center of the rod along its perpendicular bisector.



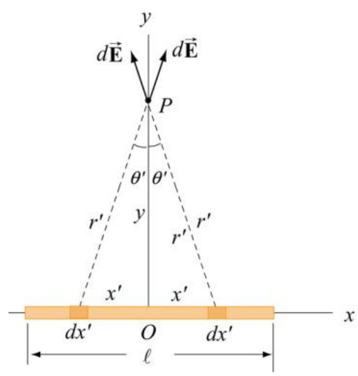
Solution:

The contribution to the electric field from a small length element $d\acute{x}$

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r'^2} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dx'}{x'^2 + y^2}$$

Where $dq=\lambda\, d\acute{x}$

The x-component of the electric field(E_x) vanishes (symmetry) as shown in figure



- The y-component of dE *is*

$$dE_y = dE\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dx'}{x'^2 + y^2}\frac{y}{\sqrt{x'^2 + y^2}} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, y\, dx'}{(x'^2 + y^2)^{3/2}}$$

- The total electric field due to the rod is

$$E_y = \int dE_y = \frac{1}{4\pi\varepsilon_0}\int_{-\ell/2}^{\ell/2}\frac{\lambda y dx'}{(x'^2 + y^2)^{3/2}} = \frac{\lambda y}{4\pi\varepsilon_0}\int_{-\ell/2}^{\ell/2}\frac{dx'}{(x'^2 + y^2)^{3/2}}$$

By making the change of variable: $x' = y\tan\theta'$, which gives $dx' = y\sec^2\theta'\, d\theta'$, the above integral becomes

$$\int_{-\ell/2}^{\ell/2}\frac{dx'}{(x'^2 + y^2)^{3/2}} = \int_{-\theta}^{\theta}\frac{y\sec^2\theta' d\theta'}{y^3(\sec^2\theta' + 1)^{3/2}} = \frac{1}{y^2}\int_{-\theta}^{\theta}\frac{\sec^2\theta' d\theta'}{(\tan^2\theta' + 1)^{3/2}} = \frac{1}{y^2}\int_{-\theta}^{\theta}\frac{\sec^2\theta' d\theta'}{\sec\theta'^3}$$

$$= \frac{1}{y^2}\int_{-\theta}^{\theta} \frac{d\theta'}{\sec\theta'} = \frac{1}{y^2}\int_{-\theta}^{\theta} \cos\theta' d\theta' = \frac{2\sin\theta}{y^2}$$

$$\boxed{E_y = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda\sin\theta}{y} = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{y}\frac{\ell/2}{\sqrt{y^2+(\ell/2)^2}}}$$

Where $sin\theta = \frac{\acute{x}}{r} = \frac{l/2}{\sqrt{y^2+(l/2)^2}}$

- In the limit where $y \gg \ell$, the above expression reduces to the "point-charge" limit:

$$E_y \approx \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{y}\frac{\ell/2}{y} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\ell}{y^2} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{y^2}$$

- On the other hand, when $\ell \gg y$, we have

$$E_y \approx \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{y}$$

- In this infinite length limit, the system has cylindrical symmetry.