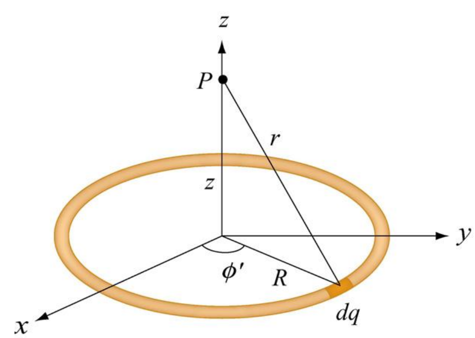
Example

Consider a uniformly charged ring of radius *R* and charge density λ. What is the electric potential at a distance *z* from the central axis?



Solution:

a small differential element $dl = R d\varphi'$ on the ring.

The element carries a charge $dq = \lambda\, dl = \lambda R\, d\varphi'$

the electric potential at *P* is

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda R\, d\phi'}{\sqrt{R^2+z^2}}$$

The electric potential at *P* due to the entire ring is

$$V = \int dV = \frac{1}{4\pi\varepsilon_0}\frac{\lambda R}{\sqrt{R^2+z^2}}\oint d\phi' = \frac{1}{4\pi\varepsilon_0}\frac{2\pi\lambda R}{\sqrt{R^2+z^2}} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{R^2+z^2}}$$

for the total charge on the ring ($Q = 2\pi R\lambda$).

In the limit $z \gg R$, the potential approaches its "point-charge" limit:

$$V \approx \frac{1}{4\pi\varepsilon_0}\frac{Q}{z}$$

the *z*-component of the electric field may be obtained as

$$E_z = -\frac{\partial V}{\partial z} = -\frac{\partial}{\partial z}\left(\frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{R^2+z^2}}\right) = \frac{1}{4\pi\varepsilon_0}\frac{Qz}{(R^2+z^2)^{3/2}}$$

Example

Consider a uniformly charged disk of radius *R* and charge density σ lying in the *xy*-plane. What is the electric potential at a distance z from the central axis?

Solution:

a circular ring of radius *r′* and width *dr′*.

The charge on the ring is

$dq = \sigma\, dA' = \sigma\,(2\pi r' dr')$

the distance from a point on the ring to *P* is

$r = (\acute{x}^2 + z^2)^{1/2}$

the electric potential at *P* is

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma(2\pi r' dr')}{\sqrt{r'^2+z^2}}$$

$$V = \frac{\sigma}{4\pi\varepsilon_0}\int_0^R \frac{2\pi r' dr'}{\sqrt{r'^2+z^2}} = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{r'^2+z^2}\right]\Bigg|_0^R = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{R^2+z^2} - |z|\right]$$ 1

In the limit $|z| \gg R$,

$$\sqrt{R^2+z^2} = |z|\left(1+\frac{R^2}{z^2}\right)^{1/2} = |z|\left(1+\frac{R^2}{2z^2}+\cdots\right)$$

the potential simplifies to the point-charge limit:

$$V \approx \frac{\sigma}{2\varepsilon_0} \cdot \frac{R^2}{2|z|} = \frac{1}{4\pi\varepsilon_0} \frac{\sigma(\pi R^2)}{|z|} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{|z|}$$

The corresponding electric field at P can be obtained as:

$$E_z = -\frac{\partial V}{\partial z} = \frac{\sigma}{2\varepsilon_0}\left[\frac{z}{|z|} - \frac{z}{\sqrt{R^2 + z^2}}\right]$$

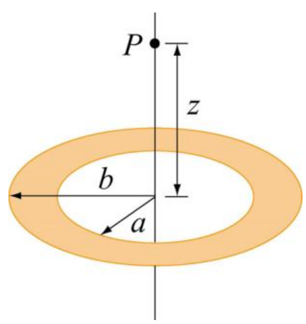
In the limit $R \gg Z$, electric field at P becomes

$$E_z = \sigma / 2\varepsilon_0$$

which is the electric field for an infinitely large non-conducting sheet.

Example

Consider an annulus of uniform charge density σ, as shown in Figure. Find the electric potential at a point P along the symmetric axis.



Solution:

A small differential element dA at a distance r away from point P is

$$dA = (\acute{r}\, d\theta) d\acute{r}$$

The amount of charge contained in dA is given by

$$dq = \sigma dA = \sigma(r' d\theta) dr'$$

the electric potential at P is

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma r' dr' d\theta}{\sqrt{r'^2 + z^2}}$$

Integrating over the entire annulus, we obtain

$$V = \frac{\sigma}{4\pi\varepsilon_0}\int_a^b \int_0^{2\pi} \frac{r' dr' d\theta}{\sqrt{r'^2 + z^2}} = \frac{2\pi\sigma}{4\pi\varepsilon_0}\int_a^b \frac{r' dr'}{\sqrt{r'^2 + z^2}} = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{b^2 + z^2} - \sqrt{a^2 + z^2}\right]$$

where we have made used of the integral

$$\int \frac{ds\, s}{\sqrt{s^2 + z^2}} = \sqrt{s^2 + z^2}$$

in the limit $a \to 0$ and $b \to R$, the potential becomes

$$V = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{R^2 + z^2} - |z|\right]$$

Example

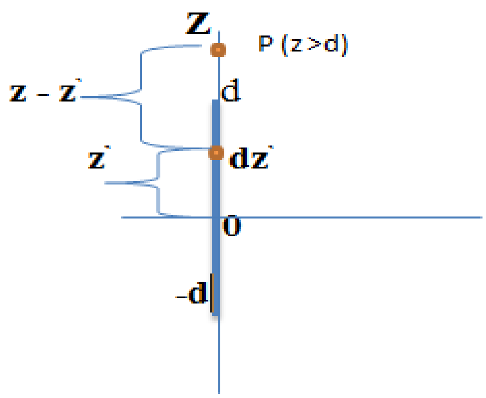
A thin rod extends along the z-axis from $z=-d$ to $z=d$. The rod carries a positive charge Q uniformly distributed along its length $2d$ *with* charge density $\lambda = Q/2d$.

(a) Calculate the electric potential at a point ($z > d$) along the z-axis.

(b) What is the change in potential energy if an electron moves from $z=4d$ to $z=3d$?

(c) If the electron started out at rest at the point $z=4d$, what is its velocity at $z=3d$?

Solution



an infinitesimal charge element $dq=\lambda\, dz'$ located at a distance $\acute{z}$ along the z-axis.

the electric potential at a point $z > d$ is

$$dV = \frac{\lambda}{4\pi\varepsilon_0}\frac{dz'}{z-z'}$$

Integrating over the entire length of the rod, we obtain

$$V(z) = \frac{\lambda}{4\pi\varepsilon_0}\int_{-d}^{+d}\frac{d\acute{z}}{z-\acute{z}} = \frac{\lambda}{4\pi\varepsilon_0} ln\left(\frac{z+d}{z-d}\right)$$

(b) Using the result derived in (a), the electrical potential at $z = 4d$ is

$$V(z=4d) = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{4d+d}{4d-d}\right) = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{5}{3}\right)$$

Similarly, the electrical potential at $z=3d$ *is*

$$V(z=3d) = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{3d+d}{3d-d}\right) = \frac{\lambda}{4\pi\varepsilon_0}\ln 2$$

The electric potential difference between the two points is

$$\Delta V = V(z=3d) - V(z=4d) = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{6}{5}\right) > 0$$

the electric potential difference ΔV is equal to the change in potential energy per unit charge, we have

$$\Delta U = q\Delta V = -\frac{|e|\lambda}{4\pi\varepsilon_0}\ln\left(\frac{6}{5}\right) < 0$$

(c) If the electron starts out at rest at $z = 4d$ then the change in kinetic energy is

$$\Delta K = \frac{1}{2}mv_f^{\,2}$$

By conservation of energy, the change in kinetic energy is

$$\Delta K = -\Delta U = \frac{|e|\lambda}{4\pi\varepsilon_0}\ln\left(\frac{6}{5}\right) > 0$$

Thus, the magnitude of the velocity at $z=3d$is

$$v_f = \sqrt{\frac{2|e|}{4\pi\varepsilon_0}\frac{\lambda}{m}\ln\left(\frac{6}{5}\right)}$$

Example:

Suppose the electric potential due to a certain charge distribution can be written in Cartesian Coordinates as

$$V(x, y, z) = Ax^2y^2 + Bxyz$$

Where A,B and C are constants. What is the associated electric field?

Solution:

The electric field can be found by using the following equation:

$$E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z}$$

$$E_x = -\frac{\partial V}{\partial x} = -2Axy^2 - Byz$$

$$E_y = -\frac{\partial V}{\partial y} = -2Ax^2y - Bxz$$

$$E_z = -\frac{\partial V}{\partial z} = -Bxy$$

Therefore, the electric field is

$$\vec{\mathbf{E}} = (-2Axy^2 - Byz)\,\hat{\mathbf{i}} - (2Ax^2y + Bxz)\,\hat{\mathbf{j}} - Bxy\,\hat{\mathbf{k}}$$

NOTE:

the electric potential at the center of the disk (z=0) is finite(by substituting z=0 into eq.1), and its value is

$$V_c = \frac{\sigma R}{2\varepsilon_0} = \frac{Q}{\pi R^2}\cdot\frac{R}{2\varepsilon_0} = \frac{1}{4\pi\varepsilon_0}\frac{2Q}{R} = 2V_0$$

This is the amount of work that needs to be done to bring a unit charge from infinity and place it at the center of the disk.